AN OPTIMAL INVESTMENT CONSUMPTION MODEL FOR RETIREES WITH NO HEALTH INSURANCE

MSc.(MATHEMATICAL SCIENCES) THESIS

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MSc. (MATHEMATICAL SCIENCES) Thesis

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Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the University or other institute of higher learning, except where due acknowledgment has been made in the text.

JOHN DAMSON MUTEPUWA

Signature

Date

CERTIFICATE OF APPROVAL

The undersigned certify that this thesis represents the student's own work and effort and has been submitted with our approval.

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ABSTRACT

Retirees meet a number of problems as they are growing older which needs persistent attention. Therefore, the decisions of the people who are close to retirement are affected undoubtedly by the consequences of financial markets. In the proposed model, stock price dynamics is assumed to follow Geometric Brownian motion (GBM) and our goal was to maximize the expected discounted utility of consumption and terminal wealth with health expenses. The investment return process comprises risk free asset and risky assets, and the health expenses. We choose power utility functions where comprehensive solutions for Hyperbolic Absolute Risk Aversion (HARA) utility functions are obtained and optimal investment, consumption and health expenditure strategies are derived by applying dynamic programming and variable change technique on the Hamilton-Jacobi-Bellman (HJB) equations. The inflation price market risk governs the amount invested in stock, bond and also how much to be put in health to sustain a given period of the retiree's lifetime. We also investigated the effects of correlation coefficients. When the value of the constant variance discounting coefficient becomes larger experienced enterprise annuity retiree reduce the proportion of their investment in the risky asset. As the health welfare rate R increases, the proportion of wealth invested in the stock becomes larger. We found that when the rate of investing in health is too high, injecting too much capital into the risky assets causes greater economic loss to the retiree's interest. Finally, a numerical example is presented to characterize the impacts of financial parameters on the optimal investment consumption strategy with health expenditure.

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LIST OF ABBREVIATIONS

CEV	Constance Elasticity of Variance
CRRA	Constant Relative Risk Aversion
CARA	Constant Absolute Risk Aversion
DC	Defined Contribution
FDM	Finite Difference Method
GBM	Geometric Brownian motion
HJB	Hamilton-Jacobi- Bellman
HCE	Health Care Expenditure
HARA	Hyperbolic Absolute Risk Aversion
IC	Investment Company
MS	Mean- Square
NESIP	National Education Sector Investment Plan
NSO	National Statistical Office
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations
PPM	Pension Planner Member
PRR	Pessimistic Return Ration
PRR	Pessimistic Return Ratio
SDE	Stochastic Differential Equation
SDDEs	Stochastic Differential Delay Equations
UNDP	United Nations Development Programme
WHO	World Health Organization

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CHAPTER 1

INTRODUCTION

1.1 Background

Retirement generally refers to the accomplishment of leaving a position of employment upon reaching a certain specified age or due to other reasons[4]. Retirement usually refers to the situation of not being paid at work. According to Atchley [4], retirement is one of the main transformations in life that make the individual leaving one part of life and joining another part of it. Usually this transition often affects many life domains and is often accompanied by a decline in life satisfaction, self-evaluation and quality of life [49, 57]. Different people think of retirement in different ways. Bur [13] looked at retirement as the situation of leaving the service voluntarily or compulsorily where such an employee has completed a specified period of service years or is removed from office by way of compulsory retirement, lay-off,dismissal (for acts of insubordination or misconduct), death, illness,disability or by voluntary withdrawal from service. Nwachukwu [56], on his part, views retirement as a socially accepted means of withdrawing from one's occupation or business in later life to enjoy leisure,freedom or simply to survive with health problems. As Jonathan Clements, a personal finance columnist with the Wall Street Journal succinctly puts it,"Retirement is like a long vacation". The goal is to enjoy it to the fullest, but not so fully that you run out of money.

Retirement can be in many forms. Bur [13] identifies various sources of retirement as including voluntary retirement, involuntary retirement, lay-offs, flexible retirement, redundancy, discharge of staff,termination of appointment and dismissal. Oyuke [57] emphasized that despite the core and importance of retirement, preparation towards it by most workers is often not sufficient. The main purpose of retirement preparation programs is to enable a worker from realistic perception of retired life and reduce anxiety about retirement. In other words, it aims at enhancing prospective retirees' adaptation to retirement and to provide assistance in managing this new phase in life [49]. As a number of young generations approach retirement, the design of products that ensure the life time financial security of retirees is at the leading edge of the agenda in the financial industry. In public policy, there is active discussions on whether the social security system can be reformed to improve the welfare of present and future retirees. Despite all these interests, little is understood about the asset allocation decisions of retirees [77]. Retirement is connected to a number of important personal decisions such as consumption, health expenditures and investment and also to policy issues such as those on insurance and pensions, as well as mandatory versus voluntary retirement [10]. The choice of the people close to retirement are affected undoubtedly by the outcomes of financial markets. For example, the big stock market boom between 1995 and 2000 led to a histrionic increase in the number of people who chose voluntary early retirement [27]. In reverse antecedent, these people's choices on retirement, consumption and savings are expected to have a notable effect on aggregate consumption and investments, and to have a large impact on the world's financial markets and economy [17].

Pension systems are very delicate issues especially in low-income countries like where most employees neither have any important retirement benefits nor earn enough during their working lives to bolster for their retirement period [2]. The lifestyle of many employees depend to a large extent on many factors, some of which include one's culture, preferences, level of resources, and the surrounding economic and social environmental factors. The issue of retirement has been alluring increasing attention in many organizations in developing countries. According to Nwachukwu [56] many factors account for this renewed emphasis, which include: No employee is expected to work throughout his/her entire life on earth; arrangements must be made for declining years. Secondly, the life expectancy of many workers has been enlarged and a majority of them are anticipated to work until they retire. Even those who believe that death could come at anytime have the aspiration that their retirement benefits would be given to their next of kin, the extended family system, which in the past helped retirees, is gradually losing its effect to western culture and influence, there is increasing awareness and emphasis that people no matter the age should learn to be self-supporting and finally the government has enacted laws motivating employers to pay retirement benefits and gratuities to certified retirees.

However, there is an argument over ever-increasing health care costs, a routine argument states that ageing of the population is a crucial driver of health care expenditure (HCE). Zweifel et al. [82](hereafter ”ZFM”) argued that this claim is a red herring. Their dissent was based on the analysis of health care expenditure of deceased persons in their last years of life. The number of quarters remaining until death was significant while the age of the persons was not. These findings have been exculpated by a number of successive studies [58]. With the reduction in mortality rate, life expectancy is continuing to increase globally. In the next $40-50$ years, the percentage of people aged over 60 will almost double all over the world. People are predicted to have longer lives and extended retirement living (WHO Report, 2005). The United Nations estimates that by 2050 there will be almost 7 billion people over 60 worldwide, close to 80% of whom will be living in low-income countries. As elsewhere, the over 60s- and particularly the over 80s-symbolize the fastest growing population group on the African continent with the numbers of aged people increasing by 50% between 2000 and 2015 and nearly fivefold by 2050 (World Economic and Social Survey, 2007). Even now one in five(an estimated over 100 million) of the world's poorest people-living on less than a dollar a day- are over 60 [66]. According to the demographic in Malawi, life expectancy at births is at 65.6 years for both sexes, 68.8 years for females and 62.5 years for males. This is a predictions per the statistics but there are some instances where individuals can reach up to 80 years. In terms of the age structure in Malawi, as of 2019, 43.47% consists of those aged $0-14$ years, 53.89% consists of those aged $15-64$ years and finally 2.64% consists of those aged 65 years and older (Malawi Population & Housing Census Report, 2018).

Retirement is one of the most significant economic events in a worker's life as such retirees receive a huge amount at retirement and are responsible for managing this wealth throughout their lives. Retirees face multiple risks including mortality, longevity, investment risks and also regulatory risks such as changes in policies and government-provided age pension entitlements which are very difficult to account for. The long term effects of this government-provided age pension entitlement systems remain unknown. Moreover, retirees and advisors have limited knowledge regarding how to consume, invest and manage the assets. This results in confusion scenarios for many retirees [1].

Therefore with the large number of individuals managing their retirement portfolio, more focus has been towards determining the optimal amount of wealth an individual should invest and consume during their retirement. The most common objectives of an individual are maintaining a certain standard of living in retirement without bankruptcy before dying and simultaneously expanding utility of consumption. Life time of an individual has significantly increased during the past decades, hence it becomes very significant for retirees to manage their portfolio of wealth proficiently in order to meet their retirement objectives [37].

According to Huang et al. [37] there are various factors that individuals should consider to successfully plan for their retirement. Some of the crucial factors are:

- Inflation rate: Funds needed to maintain a certain standard of living today will not be enough to maintain the same standard of living in the future. For instance, annual inflation rate of as low as 7% might lead to nearly triple the consumption costs for a retiree over the next 20 years.

- Longevity: Here this means living longer than expected can end up in less consumption in the near future of retirement and might bring financial ruin.

- Lifestyle: Retirees may need to maintain the same lifestyle in their retirement as they enjoyed during their working years (e.g. housing, consumption). Nevertheless, the costs involved in following the same lifestyle might bring a significant change for a retiree. For instance, an individual might want to settle down in a new country like Norway after retirement where the cost of living might be significantly very high different from their current cost of living, say in Malawi. A person might move to a smaller or bigger house for their remaining lifetime.

- Bequest motive: In this decisions to have a bequest will affect the withdrawal strategy of the retiree during their remaining lifetime. Hence, it might lead to a lower amount available for consumption and investments during their retirement.

- Health care: A retiree who is usually prone to more sickness will have to consider higher health care costs during their retirement period and therefore will consume less amount of money.

Most state and local government employers allow retirees to continue to be enrolled in their employee health plans. While retiree health insurance (RHI) is prevalent in some sectors, this benefit has been dying out in many sectors. According to Franzel and Brown [28] there are significant factors that influence firms to eliminate the RHI benefit which include the continuing rapid increase in the cost of health insurance integrated with the old age of the individuals and also increasing longevity of retirees (which results in more retirees relative to active workers and increase the total cost).

1.2 Statement of the Problem

Most people in Africa, including Malawi, retire with no medical cover and mostly old age comes with chronic illness that needs thorough and frequent attention. However, there is an increase in retirees of managing their retirement plans for the past few years. With the growing ageing population, Malawi can still face a boundless ageing problem. The possible impact may include economy sluggishness, an increase in spending due to old age which in the long run may need government to intervene on the problem.

As individual's life time can be simulated by a change in his/her health status ,it becomes very significant to conform his or her optimal consumption and investment strategies respectively. Retirees meet a number of problems as they are growing older which needs persistent attention. Therefore, the decisions of the people who are close to retirement are affected undoubtedly by the consequences of financial markets [26]. Retirement has been an issue needing attention and it has been an outstanding issue of discussion for the past decades in low-income countries. Nevertheless, there is no model for retirees that face health conditions with no health insurance. Many research papers focused much on optimal investment and consumption model based on housing, how much to borrow using the house as a collateral [19].

The model in our research is constructed to help retirees survive with no bankruptcy. In this research our model is constructed for a retiree's entire life time where we did not use the elasticity parameter. We used some economic and market parameters in other studies to evaluate the optimal investment, consumption and health expense strategies. Therefore, in our case we did not use CEV model as that of Chang [16], as such our additional contribution is to introduce the aspects of health expenses unlike in other papers, which did not focus on health expenses.

1.3 Research Objectives

1.3.1 General Objective

The main objective of the research was to examine the optimal spending and investment decisions of a retiree with no medical cover.

1.3.2 Specific Objectives

The specific objectives of this study were to:

1. Develop a wealth equation of the retiree that explains his/her financial status at any time t based on his/her investments and expenditures.

2. Transform the model into a stochastic optimal control problem based on a Hyperbolic Absolute Risk Aversion (HARA) utility function.

3. Analyse the model analytically and numerically using Hamilton- Jacobi- Bellman (HJB) equations.

1.4 Significance of the Study

Since in many countries is to nurture an investment mindset and promote access to affordable and reliable finance in keeping up with the aspiration of self-reliance and inclusive wealth creation, it will help a number of retirees to know the optimal amount of money for consumption, investment and health expenditures. The research will help retirees in prioritizing the investment and entrepreneurship development drive in order to achieve inclusive wealth creation and self-reliance. Consumption and Investment are important for growth where the division of society's resources in human capital, research and development is central to standards of living. The research will help retirees to ensure that spending is within available resources to reduce incurrence of expenditure arrears which are detrimental to the country's development agenda. Furthermore, this work will provide a platform for further research on optimal investment consumption models using different approaches.

1.5 Structure of the Thesis

The thesis comprises five chapters including this one. Chapter 2 presents a review of literature on the methodology for optimal investment consumption modeling. In Chapter 3 there is derivation of the wealth equation of the retiree at any time t that includes health expenditure and also the optimal control model which is stochastic in nature based on a Hyperbolic Absolute Risk Aversion (HARA) utility function, then a closed form solution of the Hamilton-Jacobi- Bellman equations (HJB). Chapter 4 presents the numerical analysis of the Hamilton-Jacobi- Bellman equations (HJB). Finally, in Chapter 5 the discussions, conclusions and recommendations are presented.

CHAPTER 2

LITERATURE REVIEW

In many studies, factors that affect the individuals optimal investment and consumption strategies have been incorporated. We present some works from the literature that deals with optimal investment-consumption choice of an investor who can invest in a risky and safe assets[54]. In section 2.1 there are different methods that have been used in modeling optimal investment consumption problems. A number of researchers have opted for different class of utility functions in the stochastic optimal control problems. Finally, the chapter is summarized in section 2.2.

2.1 Optimal Investment Consumption Models in Literature

The vital model of individual cost-effective and consumption was originally proposed by Fisher [29], which was later extended by Yaari [78] to include bequests in trying to find out the optimal consumption behaviour of an individual having a passivity lifetime. Yaari [78] introduces stochastic future lifetimes in the optimization problem to cogitate that the investor should adjust their optimal consumption depending on their expected survival rates. This was done by maximizing the utility function, originally proposed by Fisher [29]. The investments were being made in the bonds only.

The seminal work of Merton introduces the stock that follow a geometric Brownian motion process in the asset portfolio and establish analytical solutions for the optimal consumption and investment strategy of an individual. Moreover, he combined multiple stocks into a mutual fund (using the so called mutual fund theorem) in the optimization problem, and also establish its solution for a wide class of utility functions [51, 52]. Merton and Richard replicate the results derived using the Delong and Chen [24] methodology and perform a what-if analysis on the overall optimal consumption level, investment and wealth strategies. In addition , they introduced a modified Cox, Ingersoll, and Ross [14] CIR hereafter model to capture the stochastic force of mortality of the retiree, and thereby discover the optimal consumption and investment strategy in Delong and Chen [24].

Wang et al. [67] studied an optimal investment consumption strategy of household based on CEV model. They assumed that the theoretical price of risky asset obeys the CEV model. Their goal was to maximize the expectation of household cumulative consumption and the discounted utility of terminal wealth and to solve the optimal consumption and investment ratio using the dynamic programming principle and HJB equation. The study used logarithmic utility and isoelastic power utility function with residual utility. Wang et al. [67] obtained the analytical solution of the household investment-consumption ration by means of guessing and variable transformation. Then the influence of general parameters on the optimal ratio in the market was analyzed by numerical simulation and diagram. Similarly, Chunxiang et al. [23] studied optimal excess-of-loss reinsurance and investment problem with delay and jump-diffusion risk process under the CEV model. In their study, the insurer is allowed to purchase excess-of -loss reinsurance and invest in a financial market, where the surplus of insurer is represented by a jump-diffusion model and the financial market consists of one risk-free asset and one risky asset whose price process is governed by a CEV model. Chunxiang et al. [23] introduced the performance-related capital inflow/outflow in their study. The insurer aimed at seeking the optimal excess-of-loss reinsurance and investment strategy to maximize the expected exponential utility of the combination of terminal wealth and average performance wealth. They derived the closed-form expressions for the optimal strategy and the optimal value function by solving HJB equation and finally they provided numerical results [23]. Chang et al. [16] considered an investment and consumption problem under the Constant elasticity of variance (CEV) model and this is an extension of the original Merton's problem. In the proposed model, stock price dynamics is assumed to follow a CEV model and the goal was to maximize the expected discounted utility of consumption and terminal wealth. They applied dynamic programming principle to first obtain the Hamilton-Jacobi-Bellman (HJB) equation for the value function. Secondly, they then chose power utility and logarithmic utility for the analysis and applied the variable change technique to obtain explicit solutions to the optimal investment and consumption strategies. Finally, they provided numerical examples to analyze the effect of market parameters on the optimal investment and consumption strategies.

Gao [32] concentrated on the Constant Elasticity of Variance (CEV) model for studying the optimal investment strategy before and after retirement in a defined contribution pension plan where benefits are paid under the form of annuities. Gao[32] used Legendre transform, dual theory and variable change technique, to derive closed-form solutions for the power and exponential utility functions in two different periods (before and after retirement). Each solution contains a modified factor which reflect an investor's decision to hedge the volatility risk. For the influence of the modified factor on the optimal strategy to be investigated, he analysed the property of the modified factor. The problem was to find the optimal investment strategy for the assets over the whole life of a participant in the plan. Following Devolder et al. [25] and Xiao et al. [75], the pension liabilities after retirement are supposed to be paid in the form of an annuity whose level is guaranteed by the insurer. They splitted the problem into two periods, before and after retirement. During the period before retirement $\Big(t \in [0, T]\Big)$, the contributions can be invested in a risk-free asset and a stock. The purpose was to maximize the utility of the final wealth at retirement. The results showed that the dynamic behaviour of the modified factor for the power utility mainly depends on the time and the investor's risk aversion coefficient, whereas it only depends on the time in the exponential case. However, the work of Basimanebotlhe and Xue [9] is analogous to that of Gao [32], the difference between their work and that of Gao [32], was on the number of assets that the fund manager trade upon. They considered the risky assets to be following the Geometric Brownian motion (GBM), whilst in Gao [32], they were modeled to be of constant elasticity of variance (CEV). The other aspect that distinguishes their study from the works of other researchers was the inclusion of a certain amount of contribution, known as the supplementary contribution other than the agreed amount with the employer. The supplementary contribution is incurred by the client in order to amortize past and present imbalances in the fund that may be due to inflation, change of policies or some other factors. The CRRA utility function was considered whereby the manager aspires to maximize his/her rewards. The minimum guarantee would simultaneously acted as the solvency level to their problem. Finally, they analysed the model numerically. They found that all the portions of the stock and bonds should be replaced by cash for the investor to hit his/her maximal rewards which was evidenced through graphs. Basimanebotlhe and Xue [9] in their research paper concluded that for those investors that are less risk averse, it is important to invest more in the bond and decrease in the stock investments. Nevertheless, there are small parameters of the risk averse and it is more risky to invest in both the risky asset and the risky-free asset. They instructed the investor to give thought the cash account but may sometimes take risks with stock market though it is risky.

Optimal investment-consumption problem has been an area considered for research for the past few years. Samuelson [60] considered a discrete-time consumption-investment model with the objective of maximizing the overall expected utility of consumption. Using dynamic stochastic programming approach, he succeeded in obtaining the optimal decision for the consumption-investment model. Merton [51, 52] extended the model of Samuelson [60] to a continuous-time framework and used stochastic optimal control methodology to obtain the optimal portfolio strategy. He showed that under the assumptions of log-normal stock returns and HARA utility, the optimal proportion invested in the stock is constant. More recently, Cheung and Yang [18] studied a dynamic consumption-investment problem in a regime-switching environment. In this scenario, the price process of the risky asset was modeled as a discrete-time regime switching process and it was found out that the optimal trading strategy and the consumption strategy are consistent with our intuition in that investors should put a larger proportion of wealth in the risky asset and consume less when the underlying Markov chain is in a ”better” regime.

Many studies like that of Hulley et al. [38] explored the outcome of the age pension on consumption and investments under Constant Relative Risk Aversion (CRRA) utility while Bateman et al. [7] compared the outcome of modelling optimal consumption and investment with Hyperbolic Absolute Risk Aversion (HARA) utility against CRRA and argued that CRRA oversimplifies risk attitudes. When the age pension is introduced, the utility function indirectly becomes a HARA utility because the age pension is intended to cover basic needs like food and shelter; hence the retiree will not be able to consume only a small amount of the money. Hence, this will in turn indicates that the financial advice is non-scalable with respect to wealth and therefore fits onto the HARA utility better.

After Merton's seminal works, many researchers have studied an optimal and portfolio selection problem with various realistic constraints [51, 52]. Merton solved the portfolio optimization problem of an agent who has a Hyperbolic Absolute Risk Aversion (HARA) type of utility function [51, 52]. However, he has not considered the labor income of an agent. Park and Jang [59] studied the optimal consumption, investment and retirement strategies with negative wealth constraints. But they considered the agent whose utility function is Constant Relative Risk Aversion (CRRA). Koo et al.[42] and Shim [63] considered an optimal consumption and portfolio selection problem of an agent who has a quadratic utility function and faces a subsistence consumption constraint. However, they did not consider the agent's labor income and the negative wealth constraint and then Lim and Shin [47] and Shim [63] derived the comprehensive solutions of optimal consumption and portfolio with general utility. They used the Martingale duality approach to obtain the comprehensive solutions and demonstrate the consequences of the proportion of the wealth constraint on the optimal consumption and portfolio.

Cocco [19] developed a model of the optimal portfolio and consumption decisions of a typical investor, who receives a stream of risky labor income. In his model, Cocco [19] found out that the investor needs to decide the size of the house to buy, the amount of money needed to consume goods, the amount of money to borrow using the house as a collateral and portfolio composition among stocks and treasury bills. In addition to that, Yao and Zhang [76] investigated the model for the optimal dynamic portfolio decisions for investors who acquire housing services from renting or owning a house. Aydilek [5] developed an optimization model of retiree decisions under recursive utility with housing. Lio and Ahn [46] studied a portfolio selection problem involving an agent's realistic housing service choice, where the agent not only has to choose the size of a house to live in, but also has to select between renting and purchasing a house. In this study it was presumed that the agent participates in a continuous-time financial market in which a risky asset and risk-free asset are traded. The market was also presumed to be frictionless, meaning that, there is no tax, no transaction cost, and no limitation on financial market participation. It was also presumed that the agent receives a constant income labor. By using a dynamic programming approach, they derived an explicit solution in which they found the optimal policies for the consumption, investment, housing service, and purchasing time for a house. They also presented various numerical demonstrations showing the impacts of parameters in the financial and housing markets and the agent's preference, which visually showed the economic implications of their model. The model made an important and unique contribution because it was a groundbreaking model for the optimal time to purchase a house, which has not been studied in depth in existing mathematical portfolio optimization model.

Zhang et al. [81] studied an optimal consumption and portfolio with labour income under inflation. In their studies the life cycle of the investor is divided into two phases of retirement and employment and a stochastic optimal control model for the optimal consumption and portfolio problem with labour income was established under inflation. Then the method of stochastic control, the HJB equation of the optimal consumption and portfolio under inflation was built and thereafter an explicit expression of the valued function of the optimization problem in the case of HARA utility was obtained. Finally, Zhang et al. [81] in their research provided the economic analysis of the results through the simulation under the condition of the given parameters. Similarly, Li and Xia [45] studied the optimal investment strategy under the disordered return and random inflation. In their study they described the optimal portfolio problem when investors have partial information in the financial market and the inflation risk is taken into account, the risky asset price is suddenly changed by impacts of major events which results in the asset return disorder at a random time. Li and Xia [45] in their research applied semi-martingale and backward stochastic differential equation theory to deduce the optimal investment strategy and value process which satisfy the exponential utility maximization of terminal real wealth. Finally, numerical simulations were presented to illustrate the effects of the expected inflation rate and inflation volatility on the optimal investment strategy and to compare the asset return disorder with the normal situation [45]. However, Wu [72] studied an investment-consumption problem under inflation. Under Wu [72] the consumption price level, the prices of the available assets, and the coefficient of the power utility were presumed to be prone to the states of underlying economy modulated by a continuous- time Markovian chain [22]. The definition of admissible strategies and the verification theory corresponding to the stochastic control problem were presented. He derived the explicit expression of the optimal investment strategy and also its existence, boundedness, and feasibility of the optimal consumption were proved. Finally, Wu [72] did the mathematical and simulation analysis on how the risk aversion, the correlation coefficient between the hike and the risky asset prices, the parameters involving the hike and the coefficient of utility affected the optimal investment and consumption strategy. Johan et al. [41] developed an expected utility model for retirement behaviour in the decumulation phase of Australian retirees. They investigated the optimum housing , consumption and risky asset allocation due to age and wealth. The model was computed as a stochastic control problem and measured using the maximum likelihood method with factual data of consumption and housing from the Australian Bureau of Statistics $2009-2010$ survey.

In another study Tiro et al. [64] investigated optimal investment, consumption and portfolio choice in a framework where the pension planner member (PPM) embarks on an investment policy to cover up for some certain life targets. The aim of the pension plan manager was to optimize the anticipation of total wealth during retirement period. The investment return process comprises risk free asset and two risky assets, and the PPM benefit lies in a complete market that is constrained by the inflation rate. Closed- form solutions for constant absolute risk aversion utility functions were obtained and optimal strategies were derived by using dynamic programming on the Hamilton-Jacobi-Bellman (HJB) equations. In their studies, they presumed that preferences showed constant absolute risk aversion associated with a mean-reverting stochastic wage process.

Constant absolute risk aversion (CARA) preferences are especially significant as a basis modeling devices for coherent tractability. The study consists of risk which was only due to stock price market and the risk associated with the inflation whilst considering consumption and income as their additional optimal strategies with more assets. They discovered that the rate of return in both consumption and income follow a special distribution in statistics called the beta distribution. This is so because the consumption function and income has some analogous construction as the beta function, which has a component that causes analogous behaviour of the graphs in Tiro et al. [64] research paper. Nkeki and Nwozo [55] in their research study investigated the optimal portfolio strategies and expected wealth with stochastic cash flows under inflation protection for an investment company (IC). It was observed and found that as the market evolve, parts of the inflation-linked bond and stock portfolio values should be transferred to the bond. They also observed that the portfolio processes involved inter-temporal hedging terms that offset any shock to both the stochastic cash inflows and cash outflows.

Burggraf et al. [12] in their studies derived the Grossman's health investment model. This model explained the relationship that is there between the demand for medical care and also the individual's health status. According to Erbsland et al. [26] scientific studies that have tested the significance of Grossman's model indicate that people tend to demand more medical services if their health decreases, thereby resulting in a negative relationship between health and health care. Therefore, Burggraf et al. [12] improved the informative value of the health investment model by introducing a reworked Grossman model, which assumed a more realistic Cobb-Douglas health investment function with decreasing returns to scale. So the Grossman's standard model assumptions were kept but simply reworked the functional specifications of the health investment production function, thereby introducing uncertainty surrounding the health status of an individual in order to come closer to a real- world individual health investment problem [83]. Finally, the uncertainty of the health status of an individual, the resulting dynamic utility maximization problem is tackled by optimal stochastic control theory. Yogo [77] also studied and developed a consumption and portfolio choice model in which a retiree faces heterochthonous and stochastic loss of value of health which influenced the marginal utility of wealth as well as life expectancy. Under Yogo's model the idea was that the retiree chooses health expenditure autogenously based on his/her health, wealth and health insurance coverages and also the retiree has to make a decision in terms of assets allocation between a riskless bond, a risky asset, a real annuity and housing.

Basimanebotlhe and Xue [9] in their paper studied the optimal investment strategies in which retirees participated in a defined contribution (DC) pension fund, with the expected minimum guarantee process. The pension fund manager aspires to optimize the surplus, where his/her benefit lies in a complete financial market that is prone to inflation rate. There were only three assets of investment being used; the non risky asset and two risky assets. The dynamics of the wealth in their model considered a certain proportion of the client's salary paid as the contribution towards the pension fund and any other extra amount paid to disburse the fund. They applied stochastic optimal control method to the portfolio management problem, an HJB equation was derived. A constant risk relative aversion (CRRA) utility function was considered to obtain the closed-form solutions for the optimal investment strategies. The risk was only due to risky asset price market, and in their work they included the risk associated with inflation.

Finally, Li et al. [44] analysed an optimal portfolio and consumption problem with stochastic factor and delay over a finite time horizon. The Hamilton-Jacobi- Bellman equations was obtained after using stochastic dynamic programming. Then the optimal investment and consumption strategies for the power utility functions were solved. Chen and Milevsky [15] discussed the use of longevity insurance to hedge against longevity risk and investigated the optimal investment strategy for the retiree. Later, Milevisky and Huang [53] examined the effects of old age risk aversion on the optimal consumption strategy of an individual having destinism force of mortality and investing in the bond only. Their study proposed that an individual's optimal withdrawal rate should be varied vigorously with a change in their old age risk aversion i.e. the risk aversion parameter in the utility function. This was later extended to the stochastic force of mortality by Huang et. al. [37] where the individual has to adjust his or her optimization strategy as immediately as the information becomes available via their health status. So the GBM is of particular use for this study because this stochastic process only allows for non-negative values and the GBM assumes that the instantaneously expected rate of return is constant as such the constant instantaneous expected drift assumption of the standard Brownian process is substituted with the constant expected rate of return in the geometric brownian process [39]. However, GBM models are doubted in terms of the accuracy of the assumptions surrounding it like the normality of the distribution and the constant volatility of returns. In conclusion, the strength of the CEV model relative to the GBM which other literatures have used is that it allows for the leverage effect i.e. adding a discount factor strengthens the model [31]. Related studies for retirees to make decisions on how to allocate assets on the financial market have considered housing as a major factor which determine consumption leaving out health expenditure [19, 76, 77]. Therefore this paper wishes to fill the gap in literature.

2.2 Summary

In the chapter, models of the investment and consumption problem were reviewed as well as different approaches in finding the explicit solution of the model. Investment and consumption problem were considered under Constant Elasticity of Variance (CEV) model. The chapter also investigated a dynamic consumption-investment problem in a regime-switching environment where the price process of the risky asset was modeled as a discrete time-regime switching process. There has been a combined multiple risky assets into a mutual fund in the optimization problem and generalization of its solution for a wide class of utility functions. Legendre transform, dual theory and variable change technique were used to derive explicit solutions for the utility functions in two different periods (before and after retirement). There has been studies on the optimal investment, consumption and portfolio choice in a framework where the pension planner member (PPM) embarks on an investment policy to cover up for some certain life targets where the PPM benefit lies in a complete financial market that is constrained by the inflation rate. In some studies the dynamics of the wealth in the model considered a certain proportion of the client's salary paid as the contribution towards the pension fund and any other extra amount paid to amortize the fund. Nevertheless, as has been noted in the reviewed literature, there is one study that focused on the optimal portfolio strategies and expected wealth with stochastic cash flows under inflation protection for an investment company (IC). The IC trade on a complete diffusion model, receives a stochastic cash inflows and pays a stochastic outflows to its holder. The cash inflows are invested into a market that is characterized by a cash account, an inflation-linked risk-free asset and a risky asset. Such approaches have not taken into consideration the health expenses for the retiree with no health insurance. Finally, the Grossman's health investment model was derived.

CHAPTER 3

METHODOLOGY

In this chapter, an overview of the wealth equation of the retiree at time t are modeled as the stochastic differential equation. The method used to model the optimal investment consumption with no health insurance were presented. The stochastic optimal control problem was derived from the wealth equation of the retiree and analyzed both analytically and numerically. The retiree has to make decisions with his or her initial wealth and seeks to maximize his/her expected value of utility with rate of consumption, health expenditure, stocks(risky asset) and bond(risky-free asset) investment under HARA utility function [51, 52].

Utility functions are a special type of functions in which there is the amount of future satisfaction to be received by an investor who aspires to enlarge his/her utility of consumptions by doing different investment choices [30]. The optimal investment consumption model was transformed into a stochastic control problem where we solved for optimal consumption, optimal risky asset allocation and health expenditure using dynamic programming principle. The dynamic programming principle transforms the original stochastic optimal control problem into the problem of solving non-linear deterministic PDE's namely HJB while martingale approach has the advantage that it is in some sense more direct and more probabilistic that dynamic programming and we don't need to assume a markovian structure [6].

We considered a financial market with a risk free asset, the bank account, earning an instantaneous continuously compounded rate of return and the risky asset. In the risky investment (stock), the price process was modeled as a stochastic process [16]. The wealth process of the retiree at time t was modeled as the stochastic differential equation. The optimal consumption was formulated subject to the wealth dynamics of the retiree and the utility function of the consumption was assumed to be concave in wealth, hence the consumption function at any time t was defined by Chang et al.[16]. We assumed that the retiree finances consumption and investments with an initial wealth and the returns from the investments i.e. there is no cash flow coming in or out which is called self-financing condition. For the health expenditure, the assumption was that the marginal product of health investment is positive. We assumed that retirees are aware of their life expectancy and can take this consideration into accounts.

The model was analytically studied using the Hamilton-Jacobi- Bellman (HJB) equations and numerically using finite difference simulation. By definition, HJB equations are non-linear partial differential equations in which the value functions are the solutions [6]. Since the optimization problem was stochastic in nature, then the HJB equations became the second-order partial differential equations (PDEs) as explained by Chang et al. [16] . The model was studied under the assumption that risky asset price dynamics was driven by Geometric Brownian Motion(GBM) [3, 62].

Finally, the analytical and numerical solutions were investigated to illustrate solution of the strategy and the utility of the proposed optimization strategy (model) respectively. The retiree's decisions were solved analytically in order to obtain the exact solutions and was also estimated numerically through simulations. Change of variables technique was used to find the closed form solution of the PDEs.Change of variables is simple to use since it solves the non-linear PDE's into linear ones. We used finite difference method, C++, python in order to simulate the model using parameters that were taken from literature [32]. Finite difference method has the advantage because of its simplicity and the possibility to easily obtain higher order approximations.

3.1 Optimal Investment Consumption Mathematical Model with Health Expenditures

In this section, we give an overview on investment consumption problem. We will explain what an investment consumption model encompass and how it is expounded in the real world finance perspective. This chapter will also briefly explain the method used in this thesis to solve investment consumption model with health expenditure.

The chapter is structured as follows: In subsection 3.2 the mathematical preliminaries are provided that are very useful for the rest of the thesis. In section 3.3 the wealth equation of the retiree is described and is then mathematically formulated while in section 3.4 the model is transformed into a stochastic control problem based on a Hyperbolic Absolute Risk Aversion (HARA) utility function. Finally, in section 3.5 we provide the explicit solution using the dynamic programming approach.

3.2 Mathematical Preliminaries/ Definitions

Before looking into stochastic control processes which allow us to define optimal allocations of wealth based on certain constraints, it is necessary to define some common concepts which will be useful throughout this thesis.

Definition 3.2.1 (Probability Space)**.** *The triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space, if the following holds:*

1. $\Omega \neq \emptyset$ *is a set and* $\mathcal{F}$ *is a* $\sigma-$ *algebra (or* $\sigma-$ *field) on* Ω*. i.e. a family of subsets of* Ω *with the following properties:*

   - $\emptyset \in \mathcal{F}$

   - *If* $F \in \mathcal{F}$*, then* $\Omega \backslash F \in \mathcal{F}$

   - *If* $\mathcal{F}_i \in \mathcal{F}$ *for all* $i \in \mathbb{N}$*, then* $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$*.*

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

2. $\mathbb{P} : \mathcal{F} \to [0,1]$ *is a probability measure i.e.*

   - $\mathbb{P}(\emptyset) = 0$ *and* $\mathbb{P}(\Omega) = 1$.

   - *If* $F_i \in \mathcal{F}$ *for all* $i \in \mathbb{N}$ *are pair wise disjoint (i.e.* $F_i \cap F_j = \emptyset$ *for* $i \neq j$*), then* $\mathbb{P}\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(F_i)$

A probability space is complete if $\mathcal{F}$ *contains all subsets* G *of* Ω *with* $\mathbb{P}-$ *outer measure zero i.e. with* $\mathbb{P}^*(G) :=$*inf* $\{\mathbb{P}(F) : F \in \mathcal{F}$ *and* $G \subset F\} = 0$ *[39].*

Definition 3.2.2 (Borel $\sigma-$ Algebra)**.** *If* $\mathcal{U}$ *is a family of subsets of* Ω*, then the* σ*- algebra generated by* $\mathcal{U}$ *is* $\mathcal{F}_{\mathcal{U}} = \bigcap\{\mathcal{F} : \mathcal{F}$ *is a* σ *- Algebra of* Ω *and* $\mathcal{U} \subset \mathcal{F}\}$*.*
If $\mathcal{U}$ *is the collection of all open subsets of a topological space* Ω *(e.g* $\Omega = \mathbb{R}^d$*), then* $\mathcal{B} = \mathcal{F}_{\mathcal{U}}$ *is called the Borel* $\sigma-$ *algebra on* Ω*. The elements* $B \in \mathcal{B}$ *are called Borel sets [39].*

Definition 3.2.3 (Measurable functions, Random Variables)**.** *A function* $X : \Omega \to \mathbb{R}^d$ *is called* $\mathcal{F}-$ *measurable if* $X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$ *for all Borel sets* $B \in \mathcal{B}$*.*
If $(\Omega, \mathcal{F}, \mathbb{P})$ *is a probability space, then every* $\mathcal{F}-$ *measurable functions is called a random variable.*
Random variables $X_1, \cdots, X_n$ *are called independent if* $\mathbb{P}\left(\bigcap_{i=1}^{n} X_i^{-1}(A_i)\right) = \prod_{i=1}^{n} \mathbb{P}\left(X_i^{-1}(A_i)\right)$ *for all* $A_1, \cdots, A_n \in \mathcal{B}$*.*
If $X : \Omega \to \mathbb{R}^d$ *is any function, then the* $\sigma-$ *algebra generated by* X *is the smallest* $\sigma-$ *algebra on* Ω *containing all the sets* $X^{-1}(B)$ *for all* $B \in \mathcal{B}$ *[39].*

Definition 3.2.4 (Filtration)**.** *A filtration is a family* $\{\mathcal{F}_t : t \geq 0\}$ *of sub* $\sigma-$ *algebras of* $\mathcal{F}$ *such that* $\mathcal{F}_s \subset \mathcal{F}_t$ *for all* $t \geq s \geq 0$*.*
If $\{X_t : t \geq 0\}$ *is a family of random variables and* X_t *is* $\mathcal{F}-$ *measurable, then* $\{X_t : t \geq 0\}$ *is adapted to* $\{\mathcal{F}_t : t \geq 0\}$*.*
$\{\mathcal{F}_t : t \geq 0\}$ *is called the natural filtration of a Stochastic process* X_t *if* $\mathcal{F}_t$ *is the smallest*

$\sigma-$ algebra which contains $\mathcal{F}^{X_s}$ for all $s \in [0,t]$ i.e. $\mathcal{F}_t = \sigma\{X_s, s \in [0,t]\}$. This is the smallest filtration to which X_t is adapted [39].

Definition 3.2.5 (Adapted Process)**.** *A stochastic process X_t is said to have adapted to filtration $\left(\mathcal{F}_t, t \geq 0\right)$ if:*
$\sigma(X_t) \subseteq \mathcal{F}_t, \forall t \geq 0$. That is, X_t is an adapted process [39].

Definition 3.2.6 (Stochastic Process)**.** *Given an index set I, a stochastic process, indexed by I is a collection of random variables $\{X_\lambda, \lambda \in I\}$ on a probability space $(\Omega; \mathcal{F}; \mathbb{P})$ taking values in a set S. It can be stated as a random function X so that: $X = \{X_t : t \geq 0\} \equiv \{X(t) : t \geq 0\}$. The set S is called the state space of the process [6].*

Definition 3.2.7 (One- Dimensional Motion)**.** *Let W be a stochastic process $W = \{W(t) : t \geq 0\}$. Then W is said to be a BM process if:*

- $W(0) = 0$.
- *$\{W(t)\}$ has stationary increments:*
  $W(t) - W(s)\ N(0, t-s), 0 \leq s \leq t$.
- *$\{W(t)\}$ has independent increments:*
  $W(d) - W(c)$ is independent of $W(b) - W(a), 0 \leq a \leq b \leq c \leq d$
  Lastly, if we define a fourth condition such that:
- $W(t)\ N(0, \sigma^2 t), \forall t > 0$

This BM process is called a Wiener process. Lastly, a standard BM process is obtained upon setting $\sigma^2 = 1$. when $\sigma^2 \neq 1$ we can achieve a standard BM process by setting $Y(t) = X(t)/\sigma$ [6].

Definition 3.2.8 (n - Dimensional Brownian Motion)**.** *Taking an $n-$ dimensional set $\{X(t)_n\}, n \geq 0$ where each $X(t)_i, 0 \leq i \leq n$ is a one-dimensional BM process, then this would be called an $n-$ dimensional BM process. An implication from the definition is* *that each coordinate of $X(t)$, will be a standard BM and independent [6].*

Definition 3.2.9 (Brownian Motion with Drift)**.** *Let X be a BM process $\{X(t) : t \geq 0\}$. Then X is said to be a BM process with drift coefficient μ and volatility σ expressed as:*
$X_t := \mu t + \sigma W_t$.
Here we note that $X(t)$ $N(\mu t, \sigma^2 t)$ as the mean now varies over time [6].

Definition 3.2.10 (Geometric Brownian Motion)**.** *Let $\{W(t), t \geq 0\}$ be a standard Brownian motion process and μ and σ^2 be the drift and variance coefficients respectively. Then we say that the process $\{S(t), t \geq 0\}$ is a Geometric Brownian motion process:*
$S(t) = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$ *[39].*

Definition 3.2.11 (Martingale)**.** *Let $X(t)$ be a process $\{(X_t, \mathcal{F}_t\}_t, t > 0 \in S$ where $X_t \in \mathbb{R}$ and S is an ordered set. Then:*

- $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$
  Then: $\{(X_t, \mathcal{F}_t)\}_t$, is a martingale.

- $\mathbb{E}(X_t|\mathcal{F}_s) \leq X_s$,
  Then: $\{(X_t, \mathcal{F}_t)\}_t$, is a super-martingale.

- $\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s$,
  Then: $\{(X_t, \mathcal{F}_t)\}$, is a sub-martingale [6, 39] .

Definition 3.2.12 (Stochastic Differential Equations)**.** *A stochastic differential equation(SDE) is an equation of the form $X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s)$ which is the same as $dX_t = f(t, X_t)\, dt + g(t, X_t)\, dW_t$. The functions $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are called drift and diffusion coefficients respectively. The solution $X(t)$ is called an Ito process [6].*

Definition 3.2.13 (Admissible Strategy)**.** *An investment and consumption strategy $(\pi_t, C(t), H(t))$ is said to be admissible if the following conditions are satisfied:*

- *$\pi_t, C(t), H(t)$ is progressively measurable.*

- $\mathbb{E}\left[\int_0^T \pi_t \sigma\right] < \infty.$

- *For all initial conditions $(t_0, r_0, \pi_0, x_0) \in [0, T] \times (0, \infty)^3$, the wealth process $X(t)$ with $X(0) = x_0 > 0$ has a pathwise unique solution [6].*

Definition 3.2.14 (Admissible Control Law)**.** *A control law called admissible if:*

- *$u(t) = g(t, X(t)) \in U$ for all $t \in \mathbb{R}_+$ and $x \in \mathbb{R}^n$, where $U \subset \mathbb{R}^k$, the geometric set modeling the constraints.*

- *For any given initial point (t, x), and for any $s \in [t, T]$, the SDE*
  $dX(s) = \mu(s, X(s), g(s, X(s)))\, ds + \sigma(s, X(s), g(s, X(s)))\, dW(s),$
  $X(t) = x$
  has a unique solution.

The class of admissible control laws is denoted by $\mathcal{U}$ [6].

Definition 3.2.15 (Utility function)**.** *A Utility function $U : [0, \infty) \times (0, \infty) \to \mathbb{R}$ is continuously differentiable function for which*

1. *$U'(t, .) > 0$ for each $t \in [0, \infty)$;*

2. *$U''(t, .) < 0$ for each $t \in [0, \infty)$; and*

3. *$\lim_{x \to \infty} U'(t, x) = 0$ for each $t \in [0, \infty)$ [39].*

Definition 3.2.16 (HARA Utility function)**.** *The HARA Utility function is of the form*

$$V(C) = \frac{(1-\gamma)}{\gamma}\Big(\frac{\beta C}{1-\gamma} + \eta\Big)^{\gamma}, \tag{3.1}$$

where C is the amount of consumption (economic variable of interest) and γ, β, and η are

constants subject to the following restrictions:
$\gamma \neq 1, \beta > 0, \left(\dfrac{\beta C}{1-\gamma} + \eta\right) > 0$, *and* $\eta = 1$ *if* $\gamma \to -\infty$ *[39].*

Definition 3.2.17 (The Value function)**.** *The value function* $\Im : R_+ \times R^n \times u \to R$ *is defined by* $\Im(t,x,u) = \mathbb{E}\left[\int_t^T F(s, X_s^u, u_s)ds + \Phi(X_T^u)\right]$ *given the dynamics*
$dX_s^u = \mu(s, X_s^u, u(s, X_s^u))ds + \sigma(s, X_s^u, u(s, X_s^u))dW_s, X_t = x.$
The optimal value function $V : R_+ \times R^n \to R$ *is defined by* $V(t,x) = \underset{u\in U}{Sup}\Im(t,x,u)$*. Thus* $\Im(t,x,u)$ *is the expected utility of using the control law* u *over the time interval* $[t,T]$ *[6].*

Definition 3.2.18 (Hamilton-Jacobi-Bellman Equations)**.** *Hamilton- Jacobi - Bellman Equations are non - linear partial differential equations whose solutions are the value functions [6].*

Theorem 1 (Value Function)**.** *[6] The value function* $V(x)$ *is concave and strictly increasing. Furthermore* V *is continuous on* $[0,\infty)$*, with* $V(0) = 0$*. If* $U(C) \to M$ *as* $C \to \infty$*, with* $M < \infty$*, then* $V(x) \to \dfrac{M}{\beta}$ *as* $x \to \infty$ *[6].*

Theorem 2. *Suppose that we have two functions* $H(t,x)$ *and* $g(t,x)$*, such that*

- H *is sufficiently integrable, and solves the HJB equation*
  $\dfrac{\partial H}{\partial t}(t,x) + \underset{u\in U}{Sup}\{F(t,x,u) + A^u H(t,x)\} = 0, \forall (t,x) \in (0,T) \times R^n.$
  $H(T,x) = \Phi(x) \forall x \in R^n.$

- *The function* g *is an admissible control law.*

- *For each fixed* (t,x)*, the supremum in the expression* $\underset{u\in U}{Sup}\{F(t,x,u) + A^u H(t,x)\}$ *is attained by the choice* $u = g(t,x)$*.*

Then the following hold:

1. *The optimal value function* V *to the control problem is given by* $V(t,x) = H(t,x)$*.*

2. *There exists an optimal control law* $\mathbf{u}$*, and in fact* $\mathbf{u(t,x)} = \mathbf{g(t,x)}$ *[6].*
   For proof of this theorem refer to [6].

Lemma 1 (Hamilton - Jacobi- Bellman Equation)**.** *[6] Assume the following:*

- *there exists an optimal control process* u

- *the optimal value function* V *is regular in the sense* $V \in C^{1,2}$.

Then we have the following:

1. V *satisfies the HJB equation* $\dfrac{\partial V}{\partial t}(t,x)+$*Sup* $\{F(t,x,u) + A^u V(t,x)\} = 0, \forall (t,x) \in (0,T) \times R^n$.
   $V(T,x) = \Phi(x) \forall x \in R^n$ *where* $A^u = \sum_{i=1}^{n} \mu^u(t,x) \dfrac{\partial}{\partial x_i} + \dfrac{1}{2} \sum_{i,j=1}^{n} C_{ij}^u(t,x) \dfrac{\partial}{\partial x_i \partial x_j}$.

2. *For each* $(t,x) \in [0,T] \times R^n$ *then the supremum in the HJB equation above is attained by* $u = \mathbf{u}(t,x)$ *[6].* ***For proof of this lemma refer to [6].***

Lemma 2. *[6] A portfolio-consumption pair* (h,c) *is self-financing if and only if*

$$dV^h(t) = V^h(t) \sum_{i=1}^{N} u_i(t) \frac{dS_i(t)}{S_i(t)} - c(t)dt. \tag{3.2}$$

3.3 Model Formulation

3.3.1 Model Description

The optimal investment consumption problem explains the optimal strategy an investor can use to spend his or her money in order to maximize his or her discounted utility and minimize the risk (loss) when he or she is confronted with only few investment choices, the money can be saved in the bank account that is the risk free asset (bond), invested in the risky asset (stock market) and used for consumptions and for health expenses.

3.3.2 *Mathematical Formulation*

We consider a retiree's investment over a time interval $[0, T]$. A financial market consists of only two assets: a risk free asset which is a bond with price process B and a risk asset which is a stock with price process S [6]. The evolution of a riskless bank account value $\Big(B(t)\Big)_{t\in\mathbb{R}_+}$ is constructed from standard returns as follows,

$$\frac{B(t+dt)-B(t)}{B(t)} = rdt,$$

$$\frac{dB(t)}{B(t)} = rdt, \tag{3.3}$$

$$B'(t) = rB(t), t \geq 0$$

with the solution,

$$B(t) = B(0)e^{rt}, t \geq 0 \tag{3.4}$$

where r is the risk-free interest rate. Hence, it follows that the evolution equation for the price of the riskless bond with positive interest r is described as,

$$dB(t) = rB(t)dt, r > 0 \tag{3.5}$$

We model the risky asset price process $\Big(S(t)\Big)_{t\in\mathbb{R}_+}$ using standard returns, from the equation,

$$\frac{S(t+dt)-S(t)}{S(t)} \simeq \frac{dS(t)}{S(t)}$$

$$= \alpha dt + \sigma dW(t),\ t \geq 0 \tag{3.6}$$

The solution of the stochastic differential equation,

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t) \tag{3.7}$$

is given by,

$$S(t) = S(0)exp\Big[\sigma W(t) + \Big(\alpha - \frac{\sigma^2}{2}\Big)t\Big],\ t \geq 0 \tag{3.8}$$

Now written in a slightly sloppy form we can write the equation (3.7) as,

$$\dot{S(t)} = \Big(\alpha + \sigma \dot{W(t)}\Big)S(t) \tag{3.9}$$

where $\dot{W}$ is "white noise" i.e. the (formal) time derivative of the Wiener process. Hence, GBM is viewed as a linear ODE, with a stochastic coefficient driven by white noise [6].Therefore, the price process follows Geometric Brownian Motion, which satisfies,

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t),\ t > 0 \tag{3.10}$$

where $\alpha(> r)$ is the rate of return and σ is the volatility, $W(t)$ is the Wiener process. $\alpha > r$ because we have to guarantee that the risky asset will have the huge profit than the risk free asset. Geometric Brownian motion is one of the fundamental building blocks for modeling of asset prices. The following notations are important in deriving the retiree's model:

- $X(t)$: the wealth of the retiree at time t.
- $C(t)$: the consumption rate at time t.
- π_t : the amounts the retiree puts in stock at time t.
- β : the rate of discount.

The retiree consumes wealth $X(t)$ at a non negative rate $C(t)$ and also spends the money at a non negative rate $H(t)$. He/she distributes the remaining amount of wealth between the two assets in time t. Total wealth of the retiree is given by,

$$X(t) = \pi^0(t) + \pi^1(t). \tag{3.11}$$

where $\pi^0(t)$ is the riskless asset and $\pi^1(t)$ is the risky asset respectively at time t and $C(t), H(t)$ denote the consumption rate and health expenditure of the retiree respectively at time t.

So from equations (3.5) we have $\dfrac{dB(t)}{B(t)} = rdt$ and in equation (3.10) we have $dS(t) = S(t)(\alpha dt + \sigma dW(t)$. In particular, the retiree in this market may transfer funds from one account to the other immediately and with no costs and also may hold short positions of any size in both financial markets. Hence under these assumptions the retiree's wealth $X(t)$ at time t changes according to the following model corresponding to the strategy $\Big(\pi(t), C(t), H(t)\Big)$:

$$dX(t) = \Big(X(t) - \pi(t)\Big)\frac{dB(t)}{B(t)} + \pi_t \frac{dS(t)}{S(t)} - C(t)dt - H(t)dt. \tag{3.12}$$

After further simplications to equation (3.12) and using Lemma 2 we get,

$$dX(t) = X(t)\Big[\Big(1 - \pi^0(t)\Big)\frac{dB(t)}{B(t)} + \pi^1(t)\frac{dS(t)}{S(t)}\Big] - C(t)dt - H(t)dt \tag{3.13}$$

with initial condition $X(0) = x$ at time t and $1 - \pi^1(t) = \pi^0(t)$ is the amount of wealth invested in the risk free asset which is a bond.

We need to consider the equations (3.5) and (3.10), then equation (3.13) is transformed to the form

$$dX(t) = X(t)\left[r(t)\pi(t)[\alpha - r]dt + \pi(t)\sigma dW(t)\right] - C(t)dt - H(t)dt \tag{3.14}$$

So equation (3.14) can be simplified as,

$$dX(t) = \Big[rX(t) + \Big(\alpha - r\Big)\pi(t) - C(t) - H(t)\Big]dt + \pi(t)\sigma dW(t),\ X(0) = x(0) > 0 \tag{3.15}$$

Then equation (3.15) can be simplified further and gives,

$$dX(t) = rX(t)dt + \Big[(\alpha - r)\pi(t)\Big]dt - C(t)dt - H(t)dt + \sigma\pi(t)dW(t) \tag{3.16}$$

with initial condition $X(0) = x$. The problem is constrained by the fact that the quantities $\pi(t), H(t), C(t)$ and $X(t)$ must be all non negative and also $\pi^0(t) + \pi^1(t) = 1, \forall t \geq 0$. $X(t)$ is non negative due to the fact that the retiree can not be bankrupt.

Therefore equation (3.16) represents the wealth equation of the retiree at any time t based on the expenditures and investments. The constraint that health investment cannot be negative is standard in the health economics literature. It reflects the irreversibility of health related expenditures and the fact that health is not a traded asset.

3.4 Stochastic Optimal Control of the Model

3.4.1 Hamilton-Jacobi-Bellman-Equation

The retiree consumes $C(t)$ on the basic and other luxury needs while $H(t)$ is for the health expenditures. He/she aims to find a strategy which consists of $\pi(t)$ for stock trading, a non negative consumption rate $C(t)$, a non negative health expenditure $H(t)$ in order to maximize the expected utility,

Let $\Gamma = \Big\{\Big(\pi(t), C(t), H(t)\Big) : 0 \leq t \leq T\Big\}$ be the set of all feasible investment-consumption proportions with health expenditures. For any combination $\Big(\pi(t), C(t), H(t)\Big)$, the equation 3.16 has a special solution. Theoretically, retirees want to invest the ultimate wealth to have the greatest expected effect, namely, $\max_{u_t}\mathbb{E}\Big\{U(X_T)\Big\}$, where $U(.)$ is a concave and continuously differentiable utility function on $\Big(-\infty, \infty\Big)$. The target expected utility function is as follows:

$$\mathbb{E}\Bigg[\mu\int_0^T e^{-\beta t}U\Big(C(t), H(t)\Big)dt\Bigg] + (1-\mu)e^{-\beta T}U(X(T))\Bigg\} \tag{3.17}$$

where β is the subjective discount.

The parameter μ determines the relative importance of the intermediate consumption and the terminal wealth. Since $U(.)$ is strictly concave there exists a unique optimal trading strategy $\Big(\pi(t), C(t), H(t)\Big)$ which makes 3.16 valid [67]. For a particular feedback function π, the value function can be defined as the Markov property. Our purpose is to maximize the retiree's expected utility during his or her lifetime, where an infinite time horizon is assumed. We get the following objective function which is referred to as the value function:

$$V(x) = \underset{\Big(\pi(t),C(t),H(t)\Big)\in\mathcal{A}}{Sup} \mathbb{E}\Bigg[\mu\int_0^T e^{-\beta t}U\Big(C(t), H(t)\Big)dt\Bigg] + \Big(1-\mu\Big)e^{-\beta T}U(X_T), \beta > 0, X(t) = x \tag{3.18}$$

with boundary condition given by

$$V\Big(T, x\Big) = \Big(1-\mu\Big)e^{-\beta T}U(x), V(t, 0) = 0. \tag{3.19}$$

The value function defined by equation (3.18) is the upper definite bound of the expected utility, so it reaches the optimal value under the stochastic constraint [67].

We apply the method of dynamic programming principle in order to derive the Hamilton- Jacobi- Bellman (HJB) Equations for the value function and investigate the optimal investment, health expenditure and consumption strategy. The basic idea of the dynamic programming principle is to consider a family of optimal control problems and establish the relationships among them via the Hamilton-Jacobi- Bellman (HJB) equation. If the HJB equation is solvable , then we will obtain the desired optimal control and this is called the Verification theorem [6].

We consider the power utility function which is defined by,

$$U(x) = \frac{x^{\eta}}{\eta}, \eta < 1, \eta \neq 0. \tag{3.20}$$

Using dynamic programming principle, we get the HJB equation which is satisfied by the value function $V(x)$ for equation (3.16) as follows:

$$\frac{\partial V}{\partial t} + \underset{\left(\pi(t),C(t),H(t)\right)\in\mathcal{A}}{Sup}\left\{\mu e^{-\beta t}U\Big(C(t),H(t)\Big) + rx\frac{\partial V}{\partial x} + [(\alpha-r)\pi(t)]\frac{\partial V}{\partial x} - C(t)\frac{\partial V}{\partial x} - H(t)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\pi^2(t)\frac{\partial^2 V}{\partial x^2}\right\} = 0 \tag{3.21}$$

where,

$$V(T,x) = (1-\mu)e^{-\beta T}U(x), V(t,0) = 0. \tag{3.22}$$

Hence, after vigorous simplification equation (3.21) becomes,

$$\frac{\partial V}{\partial t} + \underset{\left(\pi(t),C(t),H(t)\right)\in\mathcal{A}}{Sup} \left\{ \mu e^{-\beta t} U\Big(C(t), H(t)\Big) + \Big[rx - C(t) + (\alpha - r)\pi(t) - H(t)\Big] \frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\pi^2(t)\frac{\partial^2 V}{\partial x^2} \right\} = 0 \tag{3.23}$$

where,

$$V(T, x) = (1-\mu)e^{-\beta T}U(x), V(t, 0) = 0 \tag{3.24}$$

and

$$\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2} \tag{3.25}$$

denote first order and second order partial derivatives with respect to the variables t and x, respectively.

3.5 The Closed-form solution

Now we need to find the explicit solution of the Hamilton-Jacobi- Bellman equation obtained in specific objective 2. We need to differentiate the HJB equation with respect to π_t.

The first-order maximizing conditions for optimal value is given by :

$$\pi^*(t) = \frac{-(\alpha - r)\dfrac{\partial V}{\partial x}}{\sigma^2 \dfrac{\partial^2 V}{\partial x^2}}. \tag{3.26}$$

Equation 3.26 can also be written in the form as shown below:

$$\pi^*(t) = \frac{-(\alpha - r)V_x}{\sigma^2 V_{xx}} \tag{3.27}$$

Therefore the first-order maximizing conditions for the optimal value is given by:

$$\begin{aligned} \pi^*(t) &= \frac{-(\alpha - r)V_x}{\sigma^2 V_{xx}}, \\ U'\Big(C^*(t), H(t)\Big) &= \frac{V_x}{\mu e^{-\beta t}}, \\ U'\Big(C(t), H(t)^*\Big) &= \frac{V_x}{\mu e^{-\beta t}}. \end{aligned} \tag{3.28}$$

Now from equation (3.28) we are differentiating the HJB equation with respect to Optimal value, the Utility function of Consumption and Utility function of Health expenditure respectively. Now introducing equation (3.28) into (3.23), we obtain,

$$\frac{\partial V}{\partial t}+\mu e^{-\beta t}U\Big(C^{*}(t),H^{*}(t)\Big)+\Bigg[rx-C^{*}(t)+-\frac{(\alpha-r)(\alpha-r)V_x}{\sigma^2 V_{xx}}-H_t^{*}\Bigg]V_x+\frac{\frac{1}{2}\sigma^2(\alpha-r)^2(V_x)^2}{\sigma^4(V_{xx})^2}\times V_{xx}=0. \tag{3.29}$$

After further and vigorous simplifications for equation (3.29), it leads to the following equation:

$$\frac{\partial V}{\partial t}+\mu e^{-\beta t}U\Bigg(C^{*}(t),H^{*}(t)\Bigg)+\Bigg[rx-C^{*}(t)-H^{*}(t)\Bigg]V_x-\frac{(\alpha-r)^2(V_x)^2}{\sigma^2 V_{xx}}+\frac{(\alpha-r)^2(V_x)^2}{2\sigma^2 V_{xx}}=0. \tag{3.30}$$

After further and vigorous simplifications in equation (3.30) we have the following equation:

$$\frac{\partial V}{\partial t}+\mu e^{-\beta t}U\Big(C^{*}(t),H^{*}(t)\Big)+\Big(rx-C^{*}(t)-H^{*}(t)\Big)V_x-\frac{1}{2}\frac{\Big(\alpha-r\Big)^2(V_x)^2}{\sigma^2 V_{xx}}=0 \tag{3.31}$$

With proper substitution where $k=\frac{1}{2}\frac{\Big(\alpha-r\Big)^2}{\sigma^2}$ we have the following equation:

$$\frac{\partial V}{\partial t}+\mu e^{-\beta t}U\Big(C^{*}(t),H^{*}(t)\Big)+\Big(rx-C^{*}(t)-H^{*}(t)\Big)V_x-K\frac{(V_x)^2}{V_{xx}}=0 \tag{3.32}$$

Here we have seen that the stochastic control problem has been transformed into a non-linear second order partial differential equation but very difficult to solve.
Now we choose the power utility function for our analysis to obtain explicit solutions to (3.31).

The power utility function is defined as :

$$U(x)=\frac{x^{\eta}}{\eta},\eta<1,\eta\neq 0. \tag{3.33}$$

For (3.31), we guess a solution with the following structure:

$$V(t,x) = e^{-\beta t}\frac{x^\eta}{\eta}f(t,x),$$
$$f(T,x) = 1-\mu. \tag{3.34}$$

Then,

$$\frac{\partial V}{\partial t} = -\beta e^{-\beta t}\frac{x^\eta}{\eta}f(t,x) + e^{-\beta t}\frac{x^\eta}{\eta}\frac{\partial f}{\partial t},$$

$$\frac{\partial V}{\partial x} = \eta e^{-\beta t}\frac{x^{\eta-1}}{\eta}f(t,x) + e^{-\beta t}\frac{x^\eta}{\eta}\frac{\partial f}{\partial x},$$

$$\frac{\partial V}{\partial x} = e^{-\beta t}x^{\eta-1}f(t,x) + e^{-\beta t}\frac{x^\eta}{\eta}\frac{\partial f}{\partial x}, \tag{3.35}$$

$$\frac{\partial^2 V}{\partial x^2} = (\eta-1)e^{-\beta t}x^{\eta-2}f(t,x) + e^{-\beta t}x^{\eta-1}\frac{\partial f}{\partial x} + e^{-\beta t}x^{\eta-1}\frac{\partial^2 f}{\partial x^2}.$$

Equation (3.35) can be re-written as follows:

$$\frac{\partial V}{\partial t} = e^{-\beta t}\frac{x^\eta}{\eta}\Big(-\beta f + f_t\Big),$$

$$\frac{\partial V}{\partial x} = e^{-\beta t}\frac{x^\eta}{\eta}\Big(\eta x^{-1}f + f_x\Big), \tag{3.36}$$

$$\frac{\partial^2 V}{\partial x^2} = e^{-\beta t}\frac{x^\eta}{\eta}\Big((\eta-1)\eta x^{-2}f + \eta x^{-1}f_x + \eta x^{-1}f_{xx}\Big).$$

Therefore equation (3.27) after thorough substitutions is rewritten as follows:

$$\pi^*(t) = \frac{-(\alpha-r)V_x}{\sigma^2 V_{xx}} \tag{3.37}$$

Substituting V_x and V_{xx} into equation (3.37) we obtain,

$$\pi^*(t) = \frac{-(\alpha - r)e^{-\beta t}\dfrac{x^\eta}{\eta}\Big(\eta x^{-1} f + f_x\Big)}{\sigma^2 e^{-\beta t}\dfrac{x^\eta}{\eta}\Big((\eta-1)\eta x^{-2} f + \eta x^{-1} f_x + \eta x^{-1} f_{xx}\Big)}. \tag{3.38}$$

After further simplifications for equation (3.38) we obtain,

$$\pi^*(t) = \frac{-(\alpha - r)\Big[\eta x^{-1} f + f_x\Big]}{\sigma^2 \Big[(\eta-1)\eta x^{-2} f + \eta x^{-1} f_x + \eta x^{-1} f_{xx}\Big]}. \tag{3.39}$$

We solve the consumption policy $C^*(t)$ and the Health investment policy $H^*(t)$. Since $U(x) = \dfrac{x^\eta}{\eta}$, then it implies that $U(C) = \dfrac{C^\eta}{\eta}$.

$$U'\Big(C^*(t)\Big) = C^{\eta-1},$$

$$U'\Big(C^*(t)\Big) = \frac{\dfrac{\partial V}{\partial x}}{\mu e^{-\beta t}},$$

$$\mu^{-1} x^{\eta-1} f(t,x) + \mu^{-1}\frac{x^\eta}{\eta}\frac{\partial f}{\partial x} = (C(t))*^{\eta-1}, \tag{3.40}$$

$$\mu^{-1} x^{\eta-1}\Big[f + \frac{1}{\eta} x f_x\Big] = (C(t))*^{\eta-1},$$

$$\mu^{\frac{1}{1-\eta}} x \Big[f + \frac{x}{\eta} f_x\Big]^{-\frac{1}{1-\eta}} = C^*(t).$$

Similarly, since $U(x) = \dfrac{x^\eta}{\eta}$, then we have,

$$U(H) = \frac{H^\eta}{\eta},$$

$$U'\Bigg(H^*(t)\Bigg) = H^{\eta-1},$$

$$U'\Bigg(C(t), H^*(t)\Bigg) = \frac{\frac{\partial V}{\partial x}}{\mu e^{-\beta t}}, \tag{3.41}$$

$$\mu^{-1}x^{\eta-1}f(t,x) + \mu^{-1}\frac{x^\eta}{\eta}\frac{\partial f}{\partial x} = (H)*^{\eta-1},$$

$$\mu^{\frac{1}{1-\eta}}x\left[f + \frac{x}{\eta}f_x\right]^{-\frac{1}{1-\eta}} = H^*(t).$$

We need to put the partial derivatives $\dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial x}, \dfrac{\partial^2 V}{\partial x^2}, U\Bigg(C^*(t), H^*(t)\Bigg), C^*(t)$ and $H^*(t)$ into equation (3.32).

Now, since

$$C^{*}(t) = \mu^{\frac{1}{1-\eta}} x \left[f + \frac{x}{\eta} f_x \right]^{\frac{-1}{1-\eta}} = H^{*}(t). \tag{3.42}$$

Hence, we need to find $U\Bigg(C^{*}(t), H(t) \Bigg)$ which will yield the same as $U\Bigg(C(t), H^{*}(t) \Bigg)$, hence it gives the following after thorough and further simplifications:

$$\begin{aligned}
U\Bigg(C^{*}(t), H(t) \Bigg) &= \frac{(C * (t)^{\eta}}{\eta}, \\
&= \frac{1}{\eta} \left(\left[\mu^{\frac{1}{1-\eta}} x f + \frac{x}{\eta} f_x \right]^{\frac{-1}{1-\eta}} \right)^{\eta}, \\
&= \frac{1}{\eta} \left[\mu^{\frac{1}{1-\eta}} x f + \frac{x}{\eta} f_x \right]^{\frac{-\eta}{1-\eta}}, \qquad (3.43) \\
&= \frac{1}{\eta} \left[\mu^{\frac{1}{1-\eta}} x f + \frac{x}{\eta} \right]^{\frac{\eta}{\eta-1}}, \\
&= U\Bigg(C(t), H^{*}(t) \Bigg)
\end{aligned}$$

We substitute the partial derivatives $\dfrac{\partial V}{\partial t}, \dfrac{\partial V}{\partial x}, \dfrac{\partial^2 V}{\partial x^2}, U\Bigg(C^*(t), H^*(t)\Bigg), C^*(t)$ and $H^*(t)$ into equation (3.32) and we obtain,

$$
\begin{aligned}
&e^{-\beta t}\frac{x^\eta}{\eta}(f_t - \beta f) + \mu e^{-\beta t}\frac{1}{\eta}\left[\mu^{\frac{1}{1-\eta}} x f + \frac{x}{\eta} f\right]^{\frac{\eta}{\eta-1}} + r x e^{-\beta t}\Bigg(x^{\eta-1} f + \frac{x^\eta}{\eta} f_x\Bigg) \\
&-\mu^{\frac{1}{1-\eta}} x\left[f+\frac{x}{\eta}f_x\right]^{-\frac{1}{1-\eta}} e^{-\beta t}\Big(x^{\eta-1}f+\frac{x^\eta}{\eta}f_x\Big)-\mu^{\frac{1}{1-\eta}} x\left[f+\frac{x}{\eta}f_x\right]^{\frac{1}{1-\eta}} e^{-\beta t}\Bigg(x^{\eta-1}f+\frac{x^\eta}{\eta}f_x\Bigg) \\
&\qquad - K\frac{\Bigg(e^{-\beta t}x^{\eta-1}f^2 + 2e^{-\beta t}x^{\eta-1} f f_x + e^{-\beta t}x^{\eta+1}\eta^{-2}(f_x)^2\Bigg)}{(\eta-1)x^{-1} f + f_x + f_{xx}} = 0 \quad (3.44)
\end{aligned}
$$

Factoring out $e^{-\beta t}\dfrac{x^\eta}{\eta}$ and simplifying equation 3.44 further we get the following:

$$
\begin{aligned}
&e^{-\beta t}\frac{x^\eta}{\eta}\Bigg[(f_t-\beta f)+x^{-\eta}\mu\Bigg(\mu^{\frac{1}{1-\eta}}xf+\frac{x}{\eta}f\Bigg)^{\frac{\eta}{\eta-1}}+rx\Bigg(\eta x^{-1}f+f_x\Bigg)-2\mu^{\frac{1}{1-\eta}}x\Bigg(\eta x^{-1}f+f_x\Bigg) \\
&\qquad \Bigg(f+\frac{x}{\eta}f_x\Bigg)^{-\frac{1}{1-\eta}} - k\frac{\Bigg(\eta x^{-1}f^2 + 2\eta x^{-1} f f_x + \frac{1}{\eta}x(f_x)^2\Bigg)}{\Bigg(\eta-1)x^{-1}f + f_x + f_{xx}\Bigg)}\Bigg] = 0 \quad (3.45)
\end{aligned}
$$

So we have,

$$
\begin{aligned}
&f_t - \beta f + \mu x^{-\eta}\Bigg(\mu^{\frac{1}{1-\eta}} x f + \frac{x}{\eta} f\Bigg)^{\frac{\eta}{\eta-1}} + rx\Bigg(\eta x^{-1} f + f_x\Bigg) - 2\mu^{\frac{1}{1-\eta}} x\Bigg(\eta x^{-1} f + f_x\Bigg) \\
&\qquad \Bigg(f + \frac{x}{\eta}f_x\Bigg)^{\frac{-1}{1-\eta}} - K\frac{\Bigg(\eta x^{-1} f^2 + 2\eta x^{-1} f_x f + \frac{1}{\eta}x(f_x)^2\Bigg)}{\Bigg((\eta-1)x^{-1} f + f_x + f_{xx}\Bigg)} = 0 \quad (3.46)
\end{aligned}
$$

Using the approach used by Gao [32] , we can apply the following power transform and variable change technique. So letting

$$f(t,x) = g(t,y), y = x^{-2\sigma} \tag{3.47}$$

We get,

$$\frac{\partial f}{\partial t} = \frac{\partial g}{\partial t},$$

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}(-2\sigma)^{-2\sigma-1}, \tag{3.48}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial y^2} 4\sigma^2 x^{-4\sigma-2} + \frac{\partial g}{\partial y}(-2\sigma)(-2\sigma-1)x^{-2\sigma-2}.$$

Introducing these derivatives (3.47) and (3.48) into (3.46) we obtain,

$$\frac{\partial g}{\partial t} - \beta g + \mu x^{-\eta}\left(\mu^{\frac{1}{1-\eta}} x g + \frac{x}{\eta} g\right)^{\frac{\eta}{\eta-1}} + rx\left(\eta x^{-1} g + \frac{\partial g}{\partial y}(-2\sigma)x^{-2\sigma-1}\right) -$$
$$2\mu^{\frac{1}{1-\eta}} x\left(\eta x^{-1} g + \frac{\partial g}{\partial y}(-2\sigma)x^{-2\sigma-1}\right)\left(g + \frac{x\partial g}{\eta\partial y}(-2\sigma)x^{-2\sigma-1}\right)^{\frac{-1}{1-\eta}} -$$
$$K\frac{\left[\eta x^{-1} g^2 + 2\eta x^{-1} g\frac{\partial g}{\partial y}(-2\sigma)x^{-2\sigma-1} + \frac{1}{\eta} x\left(\frac{\partial g}{\partial y}\right)^2 4\sigma^2 x^{-4\sigma-2}\right]}{\left[(\eta-1)x^{-1} g + \frac{\partial g}{\partial y}(-2\sigma)x^{-2\sigma-1} + \frac{\partial^2 g}{\partial y^2} 4\sigma^2 x^{-4\sigma-2} + \frac{\partial g}{\partial y}(-2\sigma)(-2\sigma-1)x^{-2\sigma-2}\right]} = 0.$$

$$g(T,y) = 1-\mu. \tag{3.49}$$

Furthermore, we use the following variable change technique. We assume that:

$$g(t,y) = h(t,y)^{1-\eta}, h(T,y) = (1-\mu)^{\frac{1}{1-\eta}}. \tag{3.50}$$

Then,

$$\frac{\partial g}{\partial t} = (1-\eta)h^{-\eta}\frac{\partial h}{\partial t},$$

$$\frac{\partial g}{\partial y} = (1-\eta)h^{-\eta}\frac{\partial h}{\partial y}, \tag{3.51}$$

$$\frac{\partial^2 g}{\partial y^2} = (1-\eta)(-\eta)h^{-\eta-1}\Big(\frac{\partial h}{\partial y}\Big)^2 + (1-\eta)h^{-\eta}\frac{\partial^2 h}{\partial y^2}.$$

In addition we need to substitute these derivatives in (3.51) into (3.49) we have,

$$\begin{aligned}(1-\eta)h^{-\eta}\frac{\partial h}{\partial t} - \beta h^{1-\eta} + \mu x^{-\eta}\left(\mu^{\frac{1}{1-\eta}}xh^{1-\eta} + \frac{x}{\eta}h^{1-\eta}\right)^{\frac{\eta}{\eta-1}} \\ + rx\left[\eta x^{-1}h^{1-\eta} + (1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1}\right] \\ - 2\mu^{\frac{1}{1-\mu}}x\left(\eta x^{-1}h^{1-\eta} + (1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1}\right) \\ \left(h^{1-\eta} + \frac{x}{\eta}(1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1}\right)^{\frac{-1}{1-\eta}} - K\frac{M}{Q} = 0.\end{aligned} \tag{3.52}$$

Hence M and Q in (3.52) represents the following:

$$M = \eta x^{-1}\Big(h^{1-\eta}\Big)^2 + 2\eta x^{-1}h^{1-\eta}(1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1} + \frac{1}{\eta}x\left((1-\eta)h^{-\eta}\frac{\partial h}{\partial y}\right)^2 4\sigma^2 x^{-4\sigma-2}. \tag{3.53}$$

and

$$Q = (\eta-1)x^{-1}h^{1-\eta} + \Big(1-\eta\Big)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1} + \Big(1-\eta\Big)(-\eta)h^{-\eta-1}\left(\frac{\partial h}{\partial y}\right)^2 + (1-\eta)h^{-\eta}\frac{\partial^2 h}{\partial y^2}4\sigma^2 x^{-4\sigma-2} + (1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)(-2\sigma-1)x^{-2\sigma-2} \tag{3.54}$$

After further and thorough simplification to equation (3.52) we get,

$$\Big(1-\eta\Big)h^{-\eta}\left[\frac{\partial h}{\partial t} + \left(\frac{r\eta-\beta}{1-\eta}\right)h + \mu x^{-\eta}\left(\mu^{\frac{1}{1-\eta}}xh^{1-\eta} + \frac{x}{\eta}h^{1-\eta}\right)^{\frac{\eta}{\eta-1}} + r x^{-2\sigma}(-2\sigma)\frac{\partial h}{\partial y} - \left(2\mu^{\frac{1}{1-\eta}}\frac{\eta h}{1-\eta} + 4\sigma\mu^{\frac{1}{1-\eta}}x^{-2\sigma}\frac{\partial h}{\partial y}\right)\left(h^{1-\eta} + \frac{x}{\eta}(1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1}\right)^{-\frac{1}{1-\eta}} - K\frac{M}{Q}\right] = 0.$$

$$h(T,y) = (1-\mu)^{\frac{1}{1-\eta}}. \tag{3.55}$$

where M and Q after further and thorough simplification yield the following respectively:

$$M = \Big(1-\eta\Big)h^{-\eta}\left[\eta x^{-1}\frac{h^{2-\eta}}{1-\eta} - 4\eta\sigma x^{-2\sigma}h^{1-\eta}\frac{\partial h}{\partial y} + \frac{1-\eta}{\eta}h^{-\eta}\left(\frac{\partial h}{\partial y}\right)^2 x^{-4\sigma-1}\right]. \tag{3.56}$$

$$Q = \Big(1-\eta\Big)h^{-\eta}\left[x^{-1}h + \frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1} - \eta h^{-1}\left(\frac{\partial h}{\partial y}\right)^2 + \frac{\partial^2 h}{\partial y^2}4\sigma^2 x^{-4\sigma-2} + \frac{\partial h}{\partial y}(-2\sigma)(2\sigma-1)x^{-2\sigma-2}\right]. \tag{3.57}$$

So we obtain the following partial differential equation:

$$\frac{\partial h}{\partial t}+\left(\frac{r\eta-\beta}{1-\eta}\right)h+\mu x^{-\eta}\left(\mu^{\frac{1}{1-\eta}}xh^{1-\eta}+\frac{x}{\eta}h^{1-\eta}\right)^{\frac{\eta}{\eta-1}}+$$
$$rx^{-2\sigma}(-2\sigma)\frac{\partial h}{\partial y}-\left(2\mu^{\frac{1}{1-\eta}}\frac{\eta h}{1-\eta}+4\sigma\mu^{\frac{1}{1-\eta}}x^{-2\sigma}\frac{\partial h}{\partial y}\right)$$
$$\left(h^{1-\eta}+\frac{x}{\eta}(1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)x^{-2\sigma-1}\right)^{-\frac{1}{1-\eta}}-K\frac{M}{Q}=0.$$
$$h(T,y)=(1-\mu)^{\frac{1}{1-\eta}}. \quad (3.58)$$

Where M and Q represent equations (3.56) and (3.57) respectively.

Noting that the equation has been a linear second order partial differential equation, it is still very difficult to solve it directly. Inspired by the approach proposed by Liu [48] we try to fit a solution to (3.58) and we have the following Lemma:

Lemma 3. *Assume that* $h(t,y)=\mu^{\frac{1}{1-\eta}}\int_t^T\tilde{h}(u,y)du+\left(1-\mu\right)^{\frac{1}{1-\eta}}\tilde{h}(t,y)$ *is a solution of (3.58); then one can prove that* $\tilde{h}(t,y)$ *satisfies the equation:*

$$\frac{\partial \tilde{h}}{\partial t}+\left(\frac{r\eta-\beta}{1-\eta}\right)\tilde{h}+\mu x^{-\eta}\left(\mu^{\frac{1}{1-\eta}}x\tilde{h}^{1-\eta}+\frac{x}{\eta}\tilde{h}^{1-\eta}\right)^{\frac{\eta}{\eta-1}}+$$
$$rx^{-2\sigma}(-2\sigma)\frac{\partial \tilde{h}}{\partial y}-\left(2\mu^{\frac{1}{1-\eta}}\frac{\eta \tilde{h}}{1-\eta}+4\sigma\mu^{\frac{1}{1-\eta}}x^{-2\sigma}\frac{\partial \tilde{h}}{\partial y}\right)$$
$$\left(\tilde{h}^{1-\eta}+\frac{x}{\eta}(1-\eta)\tilde{h}^{-\eta}\frac{\partial \tilde{h}}{\partial y}(-2\sigma)x^{-2\sigma-1}\right)^{-\frac{1}{1-\eta}}-K\frac{M}{Q}=0.$$
$$\tilde{h}\Big(T,y\Big)=1. \quad (3.59)$$

proof

Define differential operator ∇ on any function $h(t,y)$ by

$$\nabla h(t,y)$$
$$= \left(\frac{r\eta-\beta}{1-\eta}\right)\tilde{h} + \mu x^{-\eta}\left(\mu^{\frac{1}{1-\eta}}x\tilde{h}^{1-\eta} + \frac{x}{\eta}\tilde{h}^{1-\eta}\right)^{\frac{\eta}{\eta-1}} +$$
$$rx^{-2\sigma}(-2\sigma)\frac{\partial\tilde{h}}{\partial y} - \left(2\mu^{\frac{1}{1-\eta}}\frac{\eta\tilde{h}}{1-\eta} + 4\sigma\mu^{\frac{1}{1-\eta}}x^{-2\sigma}\frac{\partial\tilde{h}}{\partial y}\right)$$
$$\left(\tilde{h}^{1-\eta} + \frac{x}{\eta}(1-\eta)\tilde{h}^{-\eta}\frac{\partial\tilde{h}}{\partial y}(-2\sigma)x^{-2\sigma-1}\right)^{-\frac{1}{1-\eta}} - K\frac{M}{Q}. \quad (3.60)$$

Then (3.58) can be rewritten as,

$$\frac{\partial h(t,y)}{\partial t} + \nabla h(t,y) = 0,$$
$$(3.61)$$
$$h(T,y) = \left(1-\mu\right)^{\frac{1}{(1-\eta)}}.$$

On the other hand, we find that

$$\frac{\partial h(t,y)}{\partial t} = -\mu^{\frac{1}{(1-\eta)}}\tilde{h}(t,y) + \left(1-\mu\right)^{\frac{1}{(1-\eta)}}\frac{\partial\tilde{h}(t,y)}{\partial t}$$

$$= \mu^{\frac{1}{(1-\eta)}}\left[\int_t^T \frac{\partial\tilde{h}(u,y)}{\partial u}du - \tilde{h}(Ty)\right] + \left(1-\mu\right)^{\frac{1}{(1-\eta)}}\frac{\partial\tilde{h}(t,y)}{\partial t}, \quad (3.62)$$

$$\nabla h(t,y) = \mu^{\frac{1}{(1-\eta)}}\int_t^T \nabla\tilde{h}(u,y)du + \left(1-\mu\right)^{\frac{1}{(1-\eta)}}.\nabla\tilde{h}(t,y).$$

Further (3.58) is reduced to

$$\mu^{\frac{1}{(1-\eta)}}\left[\int_t^T\left(\frac{\partial\tilde{h}(u,y)}{\partial u}+\nabla\tilde{h}(u,y)\right)du-\tilde{h}(T,y)+1\right]+\left(1-\mu\right)^{\frac{1}{(1-\eta)}}\left[\frac{\partial\tilde{h}(t,y)}{\partial t}+\nabla\tilde{h}(t,y)\right] = 0. \quad (3.63)$$

Then we obtain

$$\frac{\partial \tilde{h}(t,y)}{\partial t} + \nabla \tilde{h}(t,y) = 0,$$

$$\tilde{h}(T,y) = 1. \tag{3.64}$$

Therefore (3.58) holds.

Taking

$$f(t,x) = g(t,y) = h\Big(t,y\Big)^{1-\eta} \tag{3.65}$$

and their relationships into considerations,

Since

$$\pi^*(t) = \frac{-(\alpha - r)\Bigg[\eta x^{-1} f + f_x\Bigg]}{\sigma^2\Bigg[(\eta-1)\eta x^{-2} f + \eta x^{-1} f_x + \eta x^{-1} f_{xx}\Bigg]}. \tag{3.66}$$

But

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y}(-2\sigma)^{-2\sigma-1}.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 g}{\partial y^2} 4\sigma^2 x^{-4\sigma-2} + \frac{\partial g}{\partial y}(-2\sigma)\Big(-2\sigma - 1\Big)x^{-2\sigma-2}.$$

$$\frac{\partial g}{\partial y} = \Big(1-\eta\Big)h^{-\eta}\frac{\partial h}{\partial y} \tag{3.67}$$

$$\frac{\partial^2 g}{\partial y^2} = \Big(1-\eta\Big)(-\eta)h^{-\eta-1}\Bigg(\frac{\partial h}{\partial y}\Bigg)^2 + \Big(1-\eta\Big)h^{-\eta}\frac{\partial^2 h}{\partial y^2}.$$

Therefore,

$$\pi^*(t) = \frac{-(\alpha - r)\left[\eta x^{-1}h^{1-\eta} + \frac{\partial g}{\partial y}(-2\sigma)^{-2\sigma-1}\right]}{\sigma^2\left[(\eta-1)\eta x^{-2}h^{1-\eta} + \eta x^{-1}\frac{\partial g}{\partial y}(-2\sigma)^{-2\sigma-1} + \eta x^{-1}\frac{\partial^2 g}{\partial y^2}4\sigma^2 x^{-4\sigma-2} + \frac{\partial g}{\partial y}(-2\sigma)(-2\sigma-1)x^{(-2\sigma-2)}\right]}. \tag{3.68}$$

After further simplifications $\pi^*(t) =$

$$\frac{-(\alpha - r)\sigma^{-2}(1-\eta)^{-1}\left[\frac{\eta}{x} + \frac{1-\eta}{h}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1}\right]}{\left[-\eta x^{-2} + \eta(xh)^{-1}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1} - \eta^2(xh^2)^{-1}\left(\frac{\partial h}{\partial y}\right)^2 + h^{-1}\frac{\partial^2 h}{\partial y^2}4\sigma^2 x^{-4\sigma-2} + h^{-1}\frac{\partial h}{\partial y}(-2\sigma)(4\sigma^2+2\sigma)x^{-2\sigma-2}\right]}.$$

Similarly,

$$C^*(t) = \mu^{\frac{1}{1-\eta}} x \left[f + \frac{x}{\eta} f_x \right]^{\frac{-1}{1-\eta}}.$$

$$= \mu^{\frac{-1}{1-\eta}} x \left[h^{1-\eta} + \frac{x}{\eta}\frac{\partial g}{\partial y}(-2\sigma)^{-2\sigma-1} \right]^{-\frac{1}{1-\eta}}.$$

$$= \mu^{\frac{1}{1-\eta}} x \left[h^{1-\eta} + \frac{x}{n}(1-\eta)h^{-\eta}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1} \right]^{-\frac{1}{1-\eta}}. \tag{3.69}$$

$$= \mu^{\frac{1}{1-\eta}} x h^{1-\eta} \left[1 + \frac{x}{\eta}(1-\eta)h^{-1}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1} \right]^{-\frac{1}{1-\eta}}.$$

Since $C^*(t) = H^*(t)$ it implies that:

$$H^*(t) = \mu^{\frac{1}{1-\eta}} x h^{1-\eta} \left[1 + \frac{x}{\eta}(1-\eta)h^{-1}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1} \right]^{-\frac{1}{1-\eta}}. \tag{3.70}$$

If utility function is given by $U(x) = \dfrac{x^\eta}{\eta}, \eta < 1$ and $\eta \neq 0$, the Optimal investment, Health investment and consumption strategy respectively of the problem Maximize $\mathbb{E}\left[\mu \int_0^T e^{-\beta t} U\bigg(C(t), H(t)\bigg) dt\right] + (1-\mu)e^{-\beta T} U(X_T)$ is given by,

$$\pi^*(t) =$$

$$\frac{-(\alpha - r)\sigma^{-2}(1-\eta)^{-1}\left[\dfrac{\eta}{x} + \dfrac{1-\eta}{h}\dfrac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1}\right]}{\left[-\eta x^{-2} + \eta(xh)^{-1}\dfrac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1} - \eta^2(xh^2)^{-1}\left(\dfrac{\partial h}{\partial y}\right)^2 + h^{-1}\dfrac{\partial^2 h}{\partial y^2}4\sigma^2 x^{-4\sigma-2} + h^{-1}\dfrac{\partial h}{\partial y}(-2\sigma)(4\sigma^2+2\sigma)x^{-2\sigma-2}\right]}.$$

$$C^*(t) = \mu^{\dfrac{1}{1-\eta}} x h^{1-\eta}\left[1 + \frac{x}{\eta}(1-\eta)h^{-1}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1}\right]^{-\dfrac{1}{1-\eta}}.$$

$$H^*(t) = \mu^{\dfrac{1}{1-\eta}} x h^{1-\eta}\left[1 + \frac{x}{\eta}(1-\eta)h^{-1}\frac{\partial h}{\partial y}(-2\sigma)^{-2\sigma-1}\right]^{-\dfrac{1}{1-\eta}}. \quad (3.71)$$

Remark: The closed form expression in equation (3.71) obtained is the general framework of the optimal investment, consumption and health strategies when stock price dynamics is given by the above stochastic processes in which it follows the geometric brownian motion. The optimal investment strategy π^* obtained has a similar form of the optimal policy under a GBM model.
According to equation (3.71), it shows that the amount of money to be spent in health will be the same as the amount of money to be spent in consumption since $H^*(t) = C^*(t)$. Schmidt and Walters [61] in their paper found out that if a person retires five years earlier at age 60, she or he expects to pay 53% more for health expenses than if he or she can wait until age 65. With such results by Schmidt and Walters [61], we can conclude in our study that a retiree have to spend 50% of his or her wealth to health and the remaining money for consumption which is closer to other studies. So in our study we assumed that the retiree have $100,000,000\$$ which he or she can use for investment, consumption and health expenditure.

3.6 Summary

In this chapter, the wealth equation of the retiree was derived in which there is inclusion of health expenditure. It was presumed that there are two types of assets in the financial market, namely risk-free assets and risky assets in which it was presumed that the price risky assets follow Geometrical Brownian Motion (GBM). Dynamic programming principle and variable change technique were applied to derive the closed-form solution to the optimal investment, consumption and health expenditure strategies in the power utility case. The basic idea of the dynamic programming was to consider a family of optimal control problems and demonstrate the relationships that exist among them via the Hamilton- Jacobi-Bellman (HJB) equation. The model developed can be used to know the optimal amount of money to be used for consumption, health expenditure and also the proportion of money to be invested in the financial market (risky asset).

CHAPTER 4

NUMERICAL ANALYSIS

In this chapter, we present the analysis by numerical simulation and graphical results to demonstrate the effect of market parameters on the optimal investment, consumption and health expenditure strategies and to give some economic implications in real world situations. An assumption is that there are two assets in the financial market, riskless asset (bond) and a risk asset which is a stock.

This chapter is organized as follows: section 4.1 describes the mathematical description and formulation of the finite difference method (FDM) for the optimal investment, consumption and health strategies. The market parameters which are taken from the literature are presented in section 4.2. In section 4.3 we provide the sensitivity analysis of market parameters on the optimal investment strategy, optimal consumption strategy and health expenditure strategy in order to give the economic implications to the real-life situations. Finally, simulation that involves GBM and SDEs is presented in section 4.4.

4.1 Mathematical Description and Formulation

In this section, we use the finite difference method (FDM) to discretize the model and simulate the results. We convert the continuous-time dynamics of the continuous-time control problem in the Hamilton-Jacobi-Bellman (HJB) into discrete-time dynamics.

A stochastic system is considered on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathcal{F}$ is a σ - algebra on $(\Omega, \mathbb{P})$ and $\mathbb{F} = \{\mathcal{F}_t | t \in T\}$ is a non-decreasing and right continuous filtration. $\mathbb{F}$ is assumed to be generated by a $d-$ dimensional standard Wiener process w with $T = [0; T]$ for some $T > 0$ and $T = [0; \infty)$ [54]. In the finite horizon case $T = [0; T]$ and assumed that an adapted process $x = \{x(t) | 0 \leq t < \infty\}$ exists, such that, at any time $t \geq 0$, the vector $x(t) \in \mathbb{R}^p$ describes the state of the system at that time, $x(t)$ is the state variable. The state variable is influenced by the choice of a feedback control $a : \mathbb{R}^p \to \mathbb{R}^q$, with $a(x)$ denoting the control applied at time t when $x(t) = x$, in such a way that x evolves according to the stochastic process

$$dx(t) = f\Big(x(t), a(x(t))\Big)dt + g\Big(x(t), a(x(t))\Big)dw(t), \tag{4.1}$$

where $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ and $g : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{p \times d}$ are continuous functions which satisfy sufficient conditions for equation (4.1) to have a solution. Define $S(x, a) = g(x, a)g(x, a)^T$ that is $S : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{p \times p}$. A feedback control a is admissible if a is progressively measurable and $a(x(s)) \in \mathbb{A} \subset \mathbb{R}^q$ for all $s \geq 0$. Furthermore, it may be required that the state variable stays within a certain subset of $\mathbb{R}^p$. With such a condition, the set of admissible controls can depend on the initial value x of the state variable [54]. Denote the set of admissible controls given x by $A(x)$. Define

$$W(x, a) = \mathbb{E}\Bigg[\int_0^\infty \exp\Bigg\{\int_0^t \beta(x(s), a(x(s)))dt\Bigg\} L(x(t), a(x(t))))dt | x(0) = x\Bigg] \tag{4.2}$$

and

$$V(x) = \underset{a \in A(x)}{Sup}\, W(x, a) \tag{4.3}$$

V is referred to as the value function. L is a continuous function satisfying a polynomial growth condition. Hence, the general optimal control problem mentioned above is solved

using much particular and specific algorithm as follows:
Consider the HJB equation associated with the control problem,

$$\underset{a\in A}{Sup}\left\{ f(x,a)^T V'(x) + \frac{1}{2}tr(S(x,a)V''(x)) + L(x,a) \leftrightarrow \beta(x,a)V(x) \right\} = 0. \tag{4.4}$$

Define a grid on the state space. Let a_0 be any admissible control. Then compute an estimate V_0 of the value function by solving numerically the PDE

$$f(x,a_0)^T V'(x) + \frac{1}{2}tr(S(x,a_0)V''(x) + L(x,a_0) \leftrightarrow \beta(x,a_0)V(x) = 0, \tag{4.5}$$

with a finite difference method. A new control is then computed as

$$a_1(x) = \underset{a\in A}{argmax}\left\{ f(x,a)^T V_0'(x) + \frac{1}{2}tr(S(x,a)V_0''(x) + L(x,a) \leftrightarrow \beta(x,a)V_0(x) \right\} = 0. \tag{4.6}$$

for all x in the grid. Then a new estimate V_1 of the value function is found by solving a PDE like equation (4.5) replacing a_0 with a_1, etc. Stability of the approach is enhanced when some upwind difference approximations are used.
Let $\delta, h > 0$ be the discrete-time step size in time and space respectively. The HJB equation is then discretized as follows:

$$V_t(t,x) = \frac{V^{\delta,h}(t+\delta,x) - V^{\delta,h}(t,x)}{\delta} + \mathcal{O}(\delta),$$

$$b^+(t,x)V_x(t,x) = b^+(t,x)\frac{V^{\delta,h}(t+\delta,x+h) - V^{\delta,h}h(t+\delta,x)}{h} + \mathcal{O}(h),$$

$$b^-(t,x)V_x(t,x) = b^-(t,x)\frac{V^{\delta,h}(t+\delta,x) - V^{\delta,h}(t+\delta,x-h)}{h} + \mathcal{O}(h),$$

$$V_{xx} = \frac{V^{\delta,h}(t+\delta,x) + V^{\delta,h}(t,x-h) - 2V^{\delta,h}(t,x)}{h^2} + \mathcal{O}(h^2),$$

where b^+ =max $\{b, 0\}$, $b^- =$ max $\{-b, 0\}$, and $b = b^+ - b^-$.

V_x is discretized differently based on its coefficient due to the following observation. Since the drift b represents the local velocity of the diffusion, b^+ and b^- are the non negative and non positive components of local velocity respectively. Therefore, b^+ pushes the diffusion located at x in a non negative direction, i.e., to x or $x - h$ [54].

Suppose we have the HJB equation of the following form for v:

$$0 = \sup_{u \in U} \left\{ V_t(t,x) + b(t,x,u(t))V_x(t,x) + \frac{1}{2}\sigma^2(t,x,u(t))V_{xx}(t,x) + U(t,u(t)) \right\}. \tag{4.7}$$

Introducing the discretizations of the HJB equations into the HJB equation (4.7) yields the difference equation

$$\begin{aligned} 0 = \sup_{u \in U} \frac{V^{\delta,h}(t+\delta,x) - V^{\delta,h}(t,x)}{\delta} + b^+(t,x)\frac{V^{\delta,h}(t+\delta,x+h) - V^{\delta,h}(t+\delta,x)}{h} \\ + b^-(t,x)\frac{V^{\delta,h}(t+\delta,x) - V^{\delta,h}(t+\delta,x-h)}{h} \\ + \frac{1}{2}\sigma^2(t,xu(t))\frac{V^{\delta,h}(t+\delta,x+h) + V^{\delta,h}(t+\delta,x-h) - 2V^{\delta,h}(t+\delta,x)}{h^2} \\ + U(t,x,u(t)), \end{aligned} \tag{4.8}$$

which can be rearranged as

$$V^{\delta,h}(t,x) = P^{x,t}_{\delta,h}(x,x+h)V^{\delta,h}(t+\delta,x+h) + P^{x,t}_{\delta,h}(x,x)V^{\delta,h}(t+\delta,x) + P^{x,t}_{\delta,h}(x,x-h)V^{\delta,h}(t+\delta,x-h) + \delta U(t,u(t)), \tag{4.9}$$

where

$$P_{\delta,h}^{x,t}(z) = \begin{cases} \dfrac{\delta}{h}(b^{+}(t,x) + \dfrac{1}{2h}\sigma^2(t,x,U(t))) \text{ if } \ z = x + h \\ \\ \dfrac{\delta}{h}(b^{-}(t,x) + \dfrac{1}{2h}\sigma^2(t,x,u(t))) \text{ if } \ z = x - h \\ \\ 1 - P_{\delta,h}^{x,t}(x+h) + P_{\delta,h}^{x,t}(x-h) \text{ if } \ z = x. \end{cases}$$

We define a time increment $\Delta t^{\delta,h} = \delta$ on the discrete state space

$$\Big\{(t,x) = (k\delta, jh) : k, j \in \mathbb{N}, 0 \leq k \leq \frac{T}{\delta}, -k \leq j \leq k\Big\} \tag{4.10}$$

by the transition probabilities $\mathbb{P}^{x,t}_{\delta,h}(z) = \begin{cases} P^{x,t}_{\delta,h}(z) \text{ if } \ z \in \{x, x+h, x-h\}, \\ \\ 0 \ \text{ otherwise} \end{cases}$

We interpret the right-hand-side of (4.9) as the dynamic principle for discrete-time state process $\xi^{\delta,h}$ and discrete-time value function $V^{\delta,h}$, under transition probability $\mathbb{P}^{x,t}_{\delta,h}$,

$$V^{\delta,h}(t,x) = \mathbb{E}^{x,t}_{\delta,h}\Big[V(t+\delta, \xi^{\delta,h})\Big] + U(t, u(t))\Delta t^{\delta,h}. \tag{4.11}$$

4.2 Market Parameters

In this section, we provide a numerical example to illustrate the effect of market parameters on the optimal investment consumption strategies under HARA utility (Power utility case). Referring to the estimation of Hong Kong stock option market by Yuen at al. [80], the parameters values are set as given in table 4.1. Without loss of generality, let $t \in [0, T]$ and $T = 5$. Throughout the analysis, unless otherwise stated, the basic parameters are given in the following table 4.1:

Table 4.1: Model Parameter Values

Symbol	**Description**	**Value**	**Source**
r	Interest rate of the bond	0.03%	Wang et al.[67]
α	Rate of return of stock	0.12%	Wang et al. [67]
σ	Volatility	-1	Wang et al. [67]
S_0	Price process	67 (million dollars)	Wang et al.[67]
t	Initial time	0 (years)	Wang et al. [67]
T	Terminal time	5 (years)	Wang et al. [67]
x_0	Initial wealth	100 (million dollars)	Assumed
η	Risk aversion	-2	Chang [16]
β	Rate of discount	0.06%	Chang [16]

4.3 Graphical Results on the Effect of Market Parameters

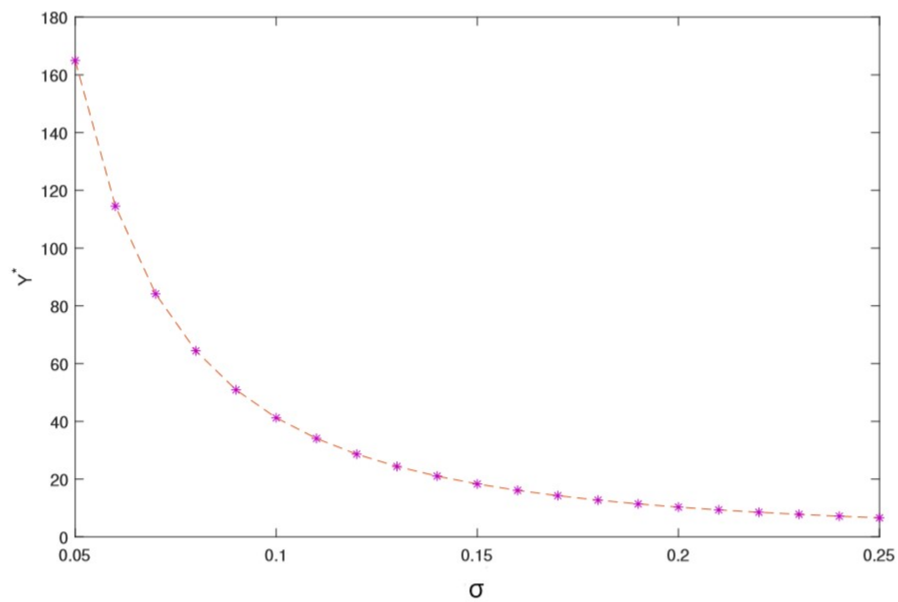

Figure 4.1: Influence of σ on optimal investment ratio Y^*

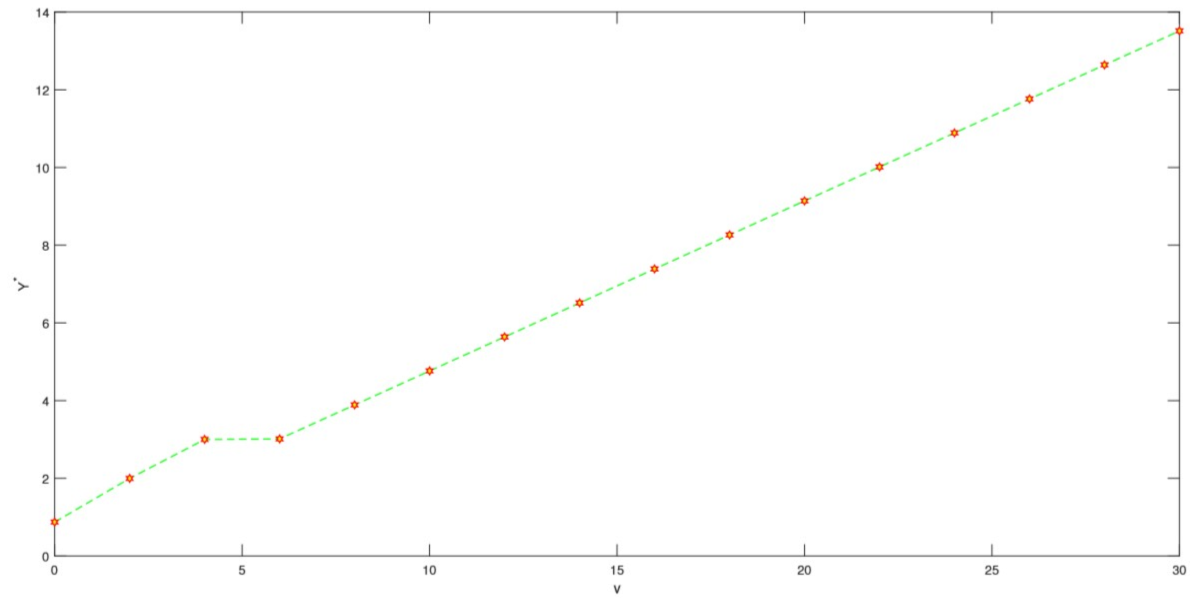

Figure 4.2: Proportional relationship between initial wealth value, health expenditure and amount invested in risky assets

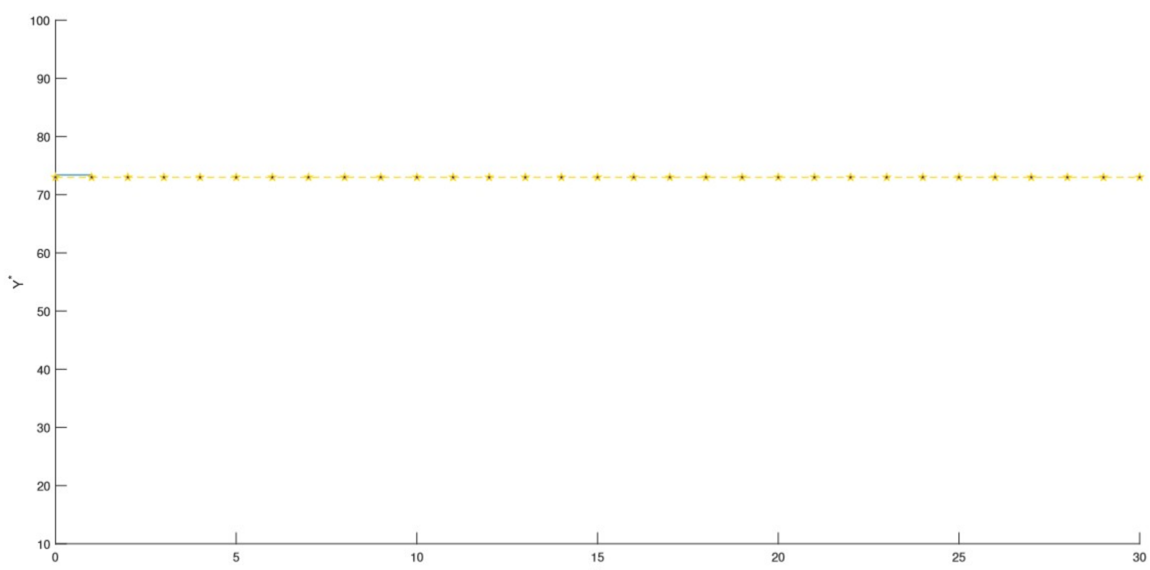

Figure 4.3: Deposit interest rate and health expense proportion

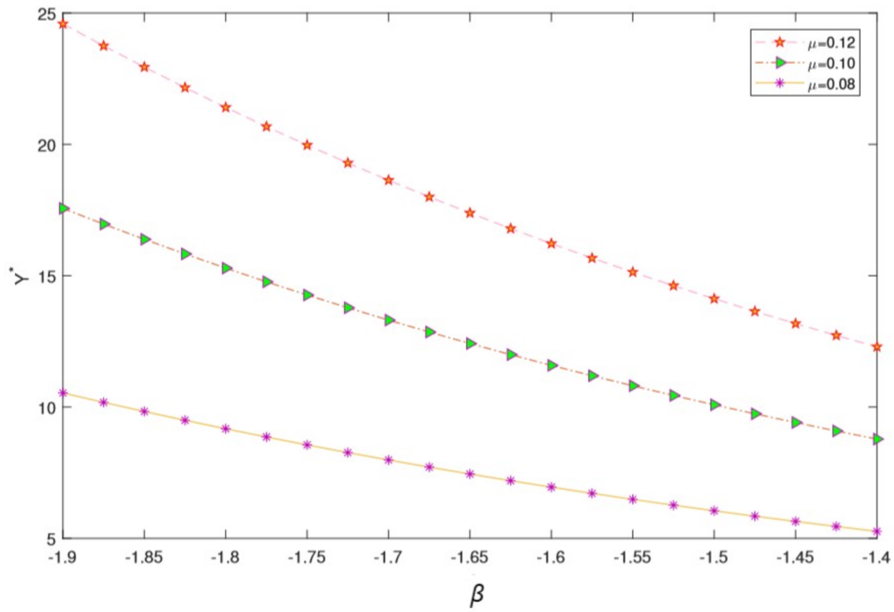

Figure 4.4: Effects of constant variance discounting coefficient and rate of return of stock on Optimal Investment strategy

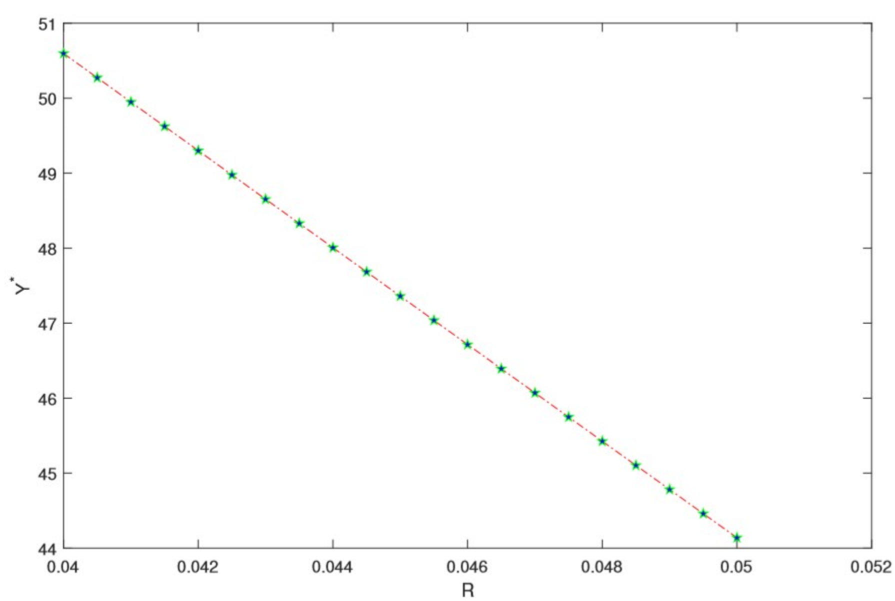

Figure 4.5: Proportion of health welfare rate R and wealth invested in stock

4.4 Sensitivity Analysis of Market Parameters

In this section, we provide the sensitivity analysis of market parameters on the optimal investment, optimal consumption and optimal health expenditures strategy respectively. We first discuss the behavioural features related to risk aversion (η), volatility(σ) and rate of discount (β) characterizing contribution into investment and health. We take the initial time $t = 0$ and assume the investor retires at $T = 5$. In the financial market, the other parameters used are $r = 0.03, \alpha = 0.12, x_0 = 100, \beta = 0.06$ and $S_0 = 67$. More results and economic implications are summarized below:

1. In figure 4.1 we compensate volatility as

$$\sigma = \alpha S_t^{\beta} \tag{4.12}$$

in which there is the optimal investment strategy Y^* of the retiree's account. Assuming the wealth value dX_t at time t is the optimal Y^*, then considering the invested risky assets and the volatility in the range $[0.05 - 0.25]$, it follows that

$$dX_t = rX(t)dt + \Big[(\alpha - r)\pi_t\Big]dt - C(t)dt - H(t)dt + \sigma\pi_t dW(t) \tag{4.13}$$

varies with it. Evidently, the increase in the parameter αS_t^{β} causes the reduction of Y^* which signifies that investors provide fewer funds to the risky asset when the stock price fluctuations increases. From a financial point of view, this can be interpreted as the stock price fluctuation enhancing the uncertainty in the market. This is in agreement with Chang [16] in his study that the optimal investment strategy is being reduced in the elasticity parameter. When the elasticity parameter becomes larger, the instantaneous volatility of the stock will be more and more bigger which leads to the more risk of investment. Therefore the retiree needs to reduce the amount invested in the stock since there is more unpredictability in this situation. In addition to that the bigger the risk aversion leads to that the investor would invest more money in the stock in order to get more income. From figure 4.1, the volatility of the stock decreases when the stock price is increasing, so the risk of investments is decreasing; hence, the investor who is the retiree has more willing to invest more money in the stock in order to get more wealth. The only difference with Chang [16] is that he used CEV model in which there is the elasticity parameter and that makes a conclusion that the amount invested in the stock under a CEV model is more than that of a GBM model. So our study have used the GBM model and thats the only difference with that of paper from Chang [16]. Furthermore to our results obtained in figure 4.1, Basimanebotlhe and Xue [9] found that the investor with higher risk aversion coefficient should invest less in the stock and the inflation linked bond at the initial stage. The real interest rate increases significantly after some more years say twelve, that would be the right time for the investor to invest more on both risky assets. They also found out that all the portions of the stock and bonds should be replaced by cash for the investor to hit his/her maximal rewards. It is better for a less risk averse investor to put in more money in the bond than in the risky assets. However, for small parameters of the risk averse parameter, it is more risky to invest in both the risky asset and the risk free asset. The mathematical and reality interpretation demonstrate how volatility affects the optimal investment strategy $(Y^*(t))$ for investing in the risky asset. The risky asset faces the possibility of depreciation. In response to this situation, a retiree as an investor does not dare to engage in costly adventures, and instead prefer conservative investments.

2. From figure 4.2, we can observe that there is a positive proportional relationship between the initial wealth $(X_0 = v)$ and the amount invested in the risky asset and in health. Thus, with the increase in the initial value of wealth, the investment of enterprise annuity retiree in the risky asset generally tends to increase. This is because the wealth of an enterprise employee is closely related to his or her ability to take risks. When the wealth value is more abundant, it has a stronger ability to resist risk. The result is that the retiree invest a large proportion of their funds in the risky assets in order to obtain larger and more satisfactory returns. But from figure 4.2 we notice that there is a zigzag somewhere which is is an indication that the retiree might not be in a good position if he or she consumes less or more since he or she is just coming from retirement, but as years are going, he or she will be able to know i.e. the more he or she is going to consume , the more he or she is going to invest. These results obtained from figure 4.2 are in agreement with Chang [16] that the increase in the initial wealth, the more amount of money invested in the risky assets and hence

health expenses is not affected. The only difference of that Chang [16] and our model is that it does not take into account transaction costs, borrowing constraints, shorting constraints and housing into the optimal investment and consumption problems resulting in a different results since our elasticity parameter is zero and a component of health expenses is included in our model making it unique.

3. As evident in figure 4.3, there is a case in which the deposit interest rate is equal to the health expense proportion. we can see the case in which investing interest rate is equal to the proportionality for the health expense rate. Compared with the second figure, in figure 4.2 it is clear that the difference between these ratings has a significant on how the retiree invest. We can easily see that the initial wealth $X_0 = v = 100$ is highly sustaining but we can consume it for instance, by taking 50% in health, invest 30% and return 20% for consumption. This is an assumption to be taken by the retiree in order not to be bankrupt. The results from the figure 4.3 in this thesis are unique. Figure 4.3 is a direct result of figure 4.2. In this case r is the expected instantaneous rate of return of the stock. Strictly R is the stocking or investing rate in risky free assets over risky assets characterized by r. The graph has that constancy property when $R = r$, otherwise graph 4.3 is graph 4.2. In Chang [16] the bigger the value of risk-free interest rate r is, the more amount invested in the risk-free asset is, and meantime the less the amount in the stock is. But the total expected wealth of the investor will become more and more larger. Comparing to this thesis, it means the amount of money the investor can spend in health will increase accordingly thereby bringing the equal proportion of health expenses and deposit interest rate. In many literatures, however, there is no results showing deposit interest rate and health expenses proportion. So our model had to take care of the health expense component thereby making the retiree to survive without bankruptcy until his or her death.

4. Figure 4.4 simply shows the effects of the elasticity coefficient β and the expected instantaneous rate of return of the stock, μ, on optimal investment strategy. From figure 4.4, we can see that the constant variance discounting coefficient and the amount invested in the risky assets are negatively correlated which tells us that when the value of the constant variance discounting coefficient becomes larger, experienced enterprise annuity retiree consciously reduce the proportion of their investment in

the risky asset. In addition, when the constant variance coefficient of the risky asset is set to a fixed value, it is found by comparison that the stock return has a certain influence on the investment ratio invested in the risky asset, and there is a positive correlation between the two. The financial background explanation is clearly apparent, the higher the expected return of the stock, the more retirees tend to invest in the risky asset. Tiro et al. [64] found that as the correlation coefficient varies in an ascending order the consumption rate and income increases. They concluded that the higher the coefficient the higher the rate of consumption and income. Correlation coefficient on consumption and income has an impact where by it governs the optimal amounts specifically from young age to late 50's. Our findings are in agreement with what other literatures have found, the only difference is the component of health expenses in our paper.

5. From figure 4.5, we see that the health expenses rate is negatively correlated with the proportion of investment in the risky asset (a retiree might choose to be inputting a proportion of investment returns in health); thus, as the health welfare rate R increases, the proportion of wealth invested in the stock becomes larger. These results are unique in our paper since many literatures did not compare the proportion of health welfare rate R and the wealth invested in stock. Their papers focused on consumption, income, housing and also people that have not yet retired. Analyzed from a financial perspective, this means that when the rate of investing in health is too high, injecting too much capital into the risky asset is very likely to cause great damage to the retiree's interests. In order to prevent greater economic losses, the retiree should avoid taking excessive risks, and instead take conservative measures. One of the conservative measure can be reducing fixed expenses (e.g., paying off a mortgage, selling a second home, throttling back support for adult children) when heading into retirement can also help reduce your reliance on investments and therefore the risk of not having enough money later.

4.5 Simulation of GBM and SDE's

In this section, we will present the analysis of the GBM and also validate the analytical solution found in the study. This section begins by presenting the simulation of GBM and see the behaviour of this model in our thesis.
We write a series of technical and analytical considerations to give the no-holds-barred low-down on optimisation for systematic optimal investment and consumption strategy.
We start with various optimization technique as follows:

- The implicit assumptions that we made when we optimized the strategy parameters.
- Best practices related to optimization of strategy parameters.
- Optimization use cases in the context of optimal investment consumption for retirees on health issues.

To make Zorro optimize a strategy's parameters, we set the parameters flag and assign calls to the optimize function to the parameters to be tuned. Whenever a parameter is optimised, it needs to be optimized towards a specific goal. The best value could be the one that maximized return, minimized drawdown, maximize sharpe ratio.
Each of these goals is likely have a different optimal parameter value. By default, Zorro uses the Pesssimistic Return Ratio(PRR) as its goal (referred to as an objective function). PRR is a gain or sustaining factor (ratio of gross wins to gross losses) penalized for lower numbers of investments and with the impact of outlying wins and losses dampened. Zorro has access to a number of investment, economic and trade, and strategy-level statistics.
We use NumWinTotal and NumLossTotal (the number of winning and losing positions respectively) to return zero if the strategy did not enter any positions. ReturnMean and ReturnStdDev are the mean and standard deviation of return on investment on a bar-by-bar basis. Thus, if the strategy is based on daily price or market bars, then ReturnMean and ReturnStdDev would be based on daily strategy returns of the retiree.

For simulating stock, asset and market prices, Geometric Brownian Motion (GBM) is the defacto go-to model. It has some nice properties which are generally consistent with stock prices, such as being log-normally distributed (and hence bounded to the downside by zero), and that expected returns dont depend on the magnitude of price.
GBM uses constant volatility. It also does not account for jumps, such as those caused by news. In spite of those limitations, GBM is useful for modeling the behaviour of stock prices. In particular, it is great for building intuition about various finance concepts- options pricing, and of course in our case we extend the idea of application to the need of asset value for retirees. When modeling these parameters, a large number of simulations are run in order to generate a distribution of possible outcomes.
Two approaches to simulating price paths using GBM:

- Using for loops to iterate over the number of price paths and the number of time-steps in each.

- Vectorisation, where we operate on an entire vector or matrix at once.

In our case, we employed the GBM and used it to estimate the distribution of asset or market value prices that influences the welfare of a retiree with health issues at some point in the future, given our model assumptions and we estimate the probability-weighted payoff curve for an option on the stock being simulated. Finally, we can get the expected value of our option by summing the area under the probability-weighted payoff curve.

4.5.1 Graphical Results on simulation of GBM and SDE's

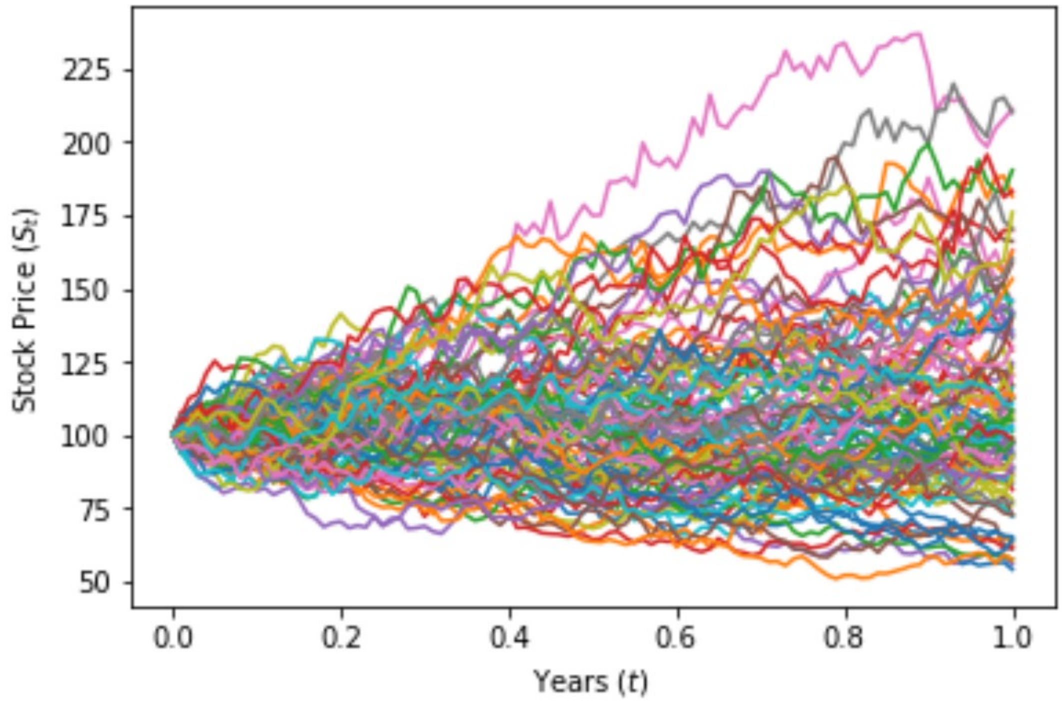

Figure 4.6: Simulation of GBM

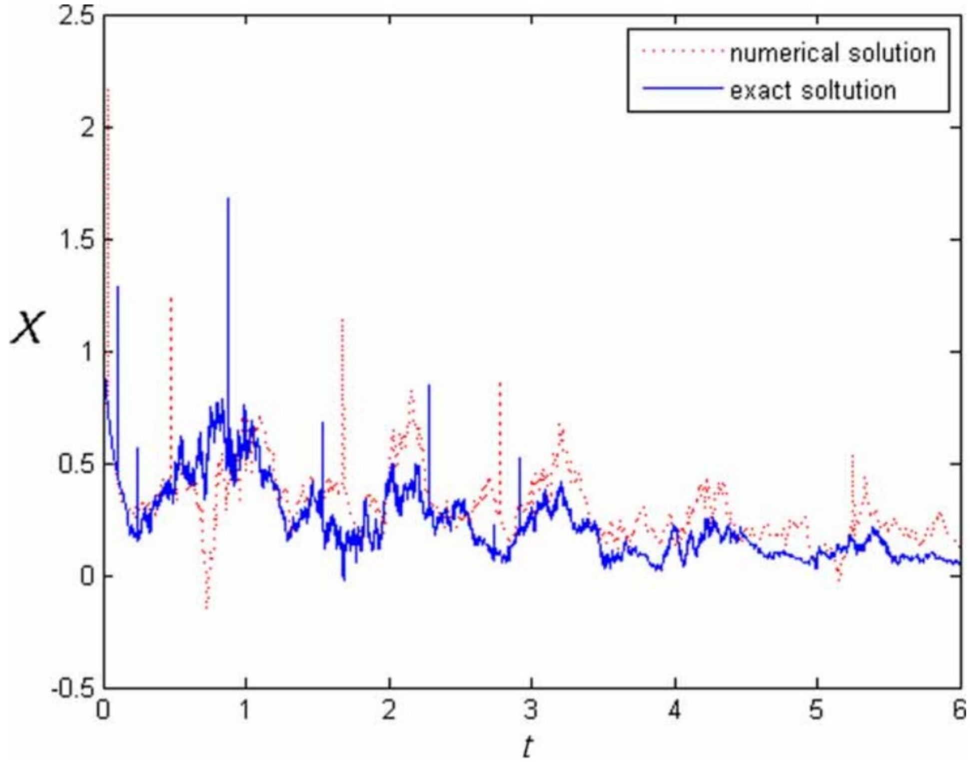

Figure 4.7: Simulation of SDE's

4.5.2 Interpretation of the simulation: GBM and SDE's

1. We define our Geometric Brownian Motion(GBM) of S which is defined by $S_0 > 0$. We need the trend of S on investment and consumption strategy of the retiree. As S varies, we define its dynamics to be modelled by,

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.14}$$

Now we write our model in an integrated form because code implementation algorithms in python need such form. Thus,

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t)$$

$$\Rightarrow \ln\Big(S(t)\Big) = \ln\Big(S(0)\Big) + \int_0^t \Big(\alpha - \frac{\sigma^2}{2}\Big)dS + \int_0^t \sigma dW_s$$

$$= \ln\Big(S(0)\Big) + \Big(\alpha - \frac{\sigma^2}{2}\Big)t + \sigma W(t) \tag{4.15}$$

$$\sim N\Bigg(\ln\Big(S(0)\Big) + \Big(\alpha - \frac{\sigma^2}{2}\Big)t, \sigma^2 t\Bigg)$$

$$\therefore S(t) = S(0)e^{\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)}$$

With the above equations GBM is simulated with respect to optimal investment strategy. Since the implementation of simulations are in python libraries, we simulate the stock price directly throughout the simulation and multiple the exponential terms together at each step. Equation 4.15 is similar to the one solved under analytical solution.

In conclusion, both the terminal pricing and expected values are characteristic measures of optimal investment consumption values for the retiree (after all, retirement benefit comprise of lifetime pension and lump sum gratuity. Qualification for pension is 10 years of service or 20 years of service in respect to early retirement. For periods of less than 5 years, an ex-gratia payment is effected. The gratuity is paid at a minimum of 5 years in service. The retirement benefit comprises of 75% of total terminal (accrued) benefits, whereas gratuity consists of the remaining 25%).

In figure (4.6), simulating stock, asset and market prices, Geometric Brownian Motion is the defacto go-to model. GBM has nice properties which are consistent with stock prices like that of log-normally distributed and that expected return dont depend on the magnitude of price. GBM is used to model the behaviour of stock prices. According to figure 4.6, GBM is used for the application of asset values for retirees. Finally, figure 4.6 is a reasonable representation of a random price described by the parameters specified in the coding in Appendices section. This is in agreement with Tiro et al. [64] explained that Wiener process modelled the random risk factor.

2. Figure 4.7 have shown a particular function which approximate a solution. A noise is removed and it shows some randomness.i.e. probability. As for $X(t)$, as time is going on and on, $X(t)$ is decaying, hence the returns is consumed stochastically. There is the numerical solution in the red dotted line and exact solution in the blue dotted line in figure 4.7. Figure 4.7 presents our SDE, and defines the mean-square (MS) stability of the Euler-Maruyama method for stochastic differential delay equations (SDDEs) with jumps. With sufficient conditions of the MS, stability of the Euler-Maruyama method for SDDEs with jumps, we can say finally we met our SDE as a class of scalar test equation that we simulated and the numerical experiments verify the results obtained from theory. Theoretically it is proved that the semi-implicit Euler method for SDEs is convergent with strong order of convergence 0.5. Therefore, we use the Euler-Maruyama method to compute an explicit solution with step-size $t = \dfrac{1}{1024}$ in our simulation. We have the numerical solution obtained from the Euler-Maruyama method with step-size $t = \dfrac{1}{128}$ together with the exact solution of the test equation parameters. The figure we have in figure 4.7 illustrates that the numerical solution converges to the exact solution and has the same stability property as its exact solution, so our theoretical interpretation tally with the numerical results. In conclusion, the model derived analytically in this study has been validated using the graphical presentation.

4.6 Summary

In this chapter, we provided the sensitivity analysis of market parameters on the optimal investment, consumption and health expenditure strategies. Hence, we had to use computational tools in solving the model. The model runs for T years over the life cycle. At every sub period t, we found optimal values for A (consumption and health share) and B (investment weight in a risky asset and perhaps risky free assets alike). The optimal values for A and B need to be maximized for the retirees' utility (power utility) every year but considered two sources of uncertainty: the return of the risky asset and the labor income stream. We had to generate a set of different scenarios (with a Monte Carlo simulation, thus for stochastic process) and we had to discretize the total wealth in order to apply a numerical dynamic programming being implemented in R, c++, python, and the matlab optimization. We solved the portfolio choice optimization problem (maximizing utility with a known utility function) in the case where all of the underlying random variables are multivariate normal. We developed an algorithm to analyse our investments using Markowits Algorithms.

CHAPTER 5

DISCUSSION, CONCLUSION, AND RECOMMENDATIONS

This chapter presents the discussion, conclusion and recommendations.

5.1 Discussion

The main aim was to examine the optimal spending and investment decisions of a retiree with no medical care. To achieve this, three specific objectives were set out as follows: developing a wealth equation of the retiree that explains his/her financial status at any time t based on his/her investments and expenditures, secondly transforming the model into a stochastic optimal control problem based on a Hyperbolic Absolute Risk Aversion (HARA) utility function and analysing the model analytically and numerically using Hamilton-Jacobi- Bellman (HJB) equations. All the three specific objectives were achieved.
In this paper, we studied optimal investment, consumption and portfolio choice in a framework where the retiree embarks on an investment policy to cover up for some certain life targets where he/she has no insurance cover. The aim here was to maximize the expectation of total wealth at the time of retirement. The investment return process comprises risk free asset and two risky assets, and the health expenses. The benefit lies in a complete market

that is constrained by the inflation rate. Closed-form solutions for hyperbolic absolute risk aversion (HARA) utility functions were obtained and optimal strategies were derived by applying dynamic programming and variable change technique on the Hamilton-Jacobi-Bellman (HJB) equations. Comparing with Chang [16], they used CEV model which is an extension of the GBM in the optimal investment and consumption strategy where all the assumptions were the same with what we used in this thesis. Unlike Tiro et al. [64] studied optimal investment, consumption and portfolio choice in a framework where the pension planner member (PPM) embarks on an investment policy to cover up for some certain life targets.

The aim of the pension plan manager was to maximize the expectation of total wealth at the time of retirement whilst in this paper we focused on health expenses to a retiree with no health insurance. In the study of Tiro et al. [64] the investment return process comprises risk free asset and two risky assets, and the PPM benefit lies in a complete market that is constrained by the inflation rate. Closed-form solutions for constant absolute risk aversion utility functions were obtained and optimal strategies were derived by applying by dynamic programming on the Hamilton-Jacobi-Bellman (HJB) equations.

Our numerical results show various effects of some economic parameters on the optimal strategies. The inflation price market risk governs the amount invested in both stock and bond. A continuous time model built a portfolio among a continuum of agents (health, assets and consumption) that influence each other strategically and have mean-variance utility function. This is the continuous time model where the nash equilibrium is built in the dynamic programming setting as a system of backward stochastic differential equations. The numerical results showed various effects of some economic and market parameters on the optimal strategies. The inflation price market risk governs the amount invested in both stock and bond, at the same time how much to be put in health to sustain a given period of the retiree's lifetime.

We also investigated the effects of high and low correlation coefficients on consumption rate and income rate. The rate of return in both consumption and income flow in health and investment followed a special distribution in statistics and probability called beta distribution because the function of consumption, investment and health income has some analogue constructions as the beta function, which has a component that causes a similar behaviour of the graphs in the thesis.

Finally, a sensitivity analysis was graphically presented to show these trends. From a financial perspective, we found out that when the rate of investing in health is too high, injecting too much capital into the risky asset is very likely to cause great damage to the retiree's interest and also the higher the expected return rate of the stock, the more retiree tend to invest in the risky assets. Our financial model fits the modern-day system of investment because it is practical for an individual to invest money only to enjoy more of it at his/her retirement age, usually after retirement.

5.2 Conclusion

Based on the results of this work, we see that the financial markets are described by the GBM. The method of stochastic optimal control and by application of the maximum principle to the optimization problem, the wealth process and admissible controls were attained. We derived the closed form solutions to the optimal investment, consumption and health expenditure strategies in the power utility case. Numerical examples were illustrated to show the effects of market parameters on the optimal investment, consumption and health expenditures and its economic implications. This study has helped retirees on much should be put in health, investment and how much to consume so that by the specified lifetime period T a retiree have something at hand before registering a zero or negative value. The amount invested will help the retiree to be compensated during the retirement period. Therefore, we conclude that a retiree should have half of his or her amount invested used for consumption, then finally, the remaining amount to be used for health expenditure.

5.3 Recommendations

Retirement takes on a new meaning as the next generation of retirees challenge age-old concepts of how to best save and prepare for the days beyond full-time employment. As a large cohort of half of the generations approach retirement, the design of products that ensure the life time financial security of retirees is at the forefront of the agenda in the financial industry. Retirement is one of the most important economic events in a worker's life and it needs to be properly planned. Not surprisingly, retirement is connected to a number of important personal decisions such as consumption and investment and also to policy issues such as those on insurance and pensions, as well as mandatory versus voluntary retirement. An increasing number of people are approaching retirement as population ageing is progressing rapidly both in the developed and the developing world. The decisions of the people close to retirement are affected significantly by the outcomes of financial markets. Thus, from the results of this work, it is recommended that;

1. As the inflation market risk increases, the retiree will have to put more money in the stock market in order to get more wealth which will help the retiree to survive by consuming and spending in health. The increase in initial wealth will improve the living standards of the retiree throughout his/her entire life.

2. When the rate of investing in health is too high, injecting too much capital into the risky assets is very likely to cause greater economic loss to the retiree's interest like greater economic loss. Therefore it is encouraged to the retiree to avoid taking excessive risks and instead take conservative measures e.g. paying off a mortgage, selling a second home etc.

3. Retirees are encouraged to reduce the investment amount in the stock to avoid risk if the risk-free interest rate is increasing since this will make his income more and survival will be possible at all cost.

4. Retirees are encouraged to only choose those investments which will bring the highest returns in order to sustain their day-to -day living.

5.4 Future Work

In light of the perpetual global economic crises which are affecting most countries, there is a need to spend within the available resources. Therefore in order to achieve an inclusive wealth creation and self-reliance, it is imperative to have comprehensive researches done. Based on the model of this study, it is proposed that future work should consider:

1. On the investment and consumption problems there is a need to focus on the cases of

more sophisticated situations such as introducing transaction costs, stochastic affine interest rate and the other uncertain factors which would result in more sophisticated nonlinear second order partial differential equations. This will help to have a complete results in a financial perspective since our model did not consider that.

2. Infinite series expansion method using power utility function. This is because further modifications on the conditions of utility function may lead to a simpler derivation of the HJB equation and this will fill the gap in this study.

3. It would be very important to investigate presumptions about preferences and constraints that are significant to explain how the assets in the financial market can be allocated and also how can health expenditure respond to health shocks on different perspectives.

4. The model can be extended by incorporating CEV model. This model is an extension of GBM and with the same assumptions there is a need to compare results of that of GBM and CEV models.

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APPENDICES

A. CODES THAT WERE USED TO SIMULATE THE OPTIMAL INVESTMENT CONSUMPTION MODEL FOR RETIREES

Implementation of HJB in C/C++

```
%----------------------------------------------------------------------------------
%----------------------------------------------------------------------------------
%Filename: HJ2DFunc.cpp

Version: 1.0
Created: 2022
Revision: none
Compiler: gcc/intel compiler

Author: John Mutepuwa
        University of Malawi
 %---------------------------------------------------------------------------------
 %---------------------------------------------------------------------------------
 /
 #include "HJ2DFunc>h"

 double func_F_V8::BETA = 1.;
```

```
double func_G_V2::T_MAX= 1.;

double func_exp::BETA = 0.25;

//double func_sin::K = M_PI;
//double func_sin::B = 0;
```

Simulation of Brownian Motion


```
%-----------------------------------------------------------------------------------
%-----------------------------------------------------------------------------------
#calculate each time step
dt = T/n
#simualtion using numpy arrays
St = np. exp(
(alpha - sigma**2/2) * dt + sigma * np. random. normal (0, np. sqrt(dt), size = (M, n)). T)
#Include array of 1's
St = np. vstack ([np.ones (M), St])
#multiply through by S0 and return the cumulative product of elemnts along a
given simulation path (axis = 0)
St = S0 * St. cumprod (axism = 0)
```

```
#define time interval correctly
time = np.linspace(0, T, n + 1)
#require numpy array that is same shape as St
tt = np.full(shape = (M , n + 1), fill_ value = timea). T
plt.plot (tt, St)
plt. xlabel (''Years $(t)$'')
plt.ylabel (''Stock Price $(S_t)$)
plt.show()
%---------------------------------------------------------------------------------
```



```
%--------------------------------------------------------------------------------
%--------------------------------------------------------------------------------
Pseudocode:
(Using Quadprog in Matlab):


% data as column based price matrix
%data = xlsread('excel_sheet.xlsx');

nAssets = size(data, 2);
```

```
rets = data(2:end, :)./data(1:end-1,:)-1;

%annualize the returns and covariance
mu = 250 * mean(rets);
sigma = 250 * cov(rets);

%formulate the problem/optimization
r_target = 0.10; %r_target is the required return
f = zeros(nAssets, 1);    %there is no constant
A = [-mu; -eye(nAssets)]; %besides the returns we forbid short selling
b = [-r_target; zeros(nAssets, 1)]; % required return and weights greater/equals 0
Aeq = ones(1, nAssets);   %All weights should sum up...
beq = 1;                  %... to one (1)

%solve the optimization
w = quadprog(sigma,f,A,b,Aeq,beq);

%print the solution
fprintf(2, 'Risk: %.3f%%\n', sqrt(w'*sigma*w)*100);
fprintf(2, 'Ret: %.3f%%\n', w'*mu'*100);
%---------------------------------------------------------------------------------
%---------------------------------------------------------------------------------
```

Computations of rates of expenses, consumption and returns

```
%-----------------------------------------------------------------------------------
%-----------------------------------------------------------------------------------

% The virtual base class defined below contains enough information
to update a retiree or an investment portfolio analyst,
 but doesn't know how to compute rates and parameters:

# include <vector>
double std :: vector<std :: size t> std :: vector<std :: size t> std ::
vector<int> rate const ;

 // from the SDE X(t) DependsOn ; marketParameters ; sigma ;

// A class simulation constructor : derived classes //
 should define their own constructors that //
  set up the various vectors above all requiring to
   analyse the effect of parameters on X(t) = InvestoConsume.

InvestoConsume( void )
```

```
{rate const = 0.0 ; double Return; rates /= parameters;
 int n = number_of_parameters// Assumed with the data,
 one can make the appropriate changes in parameter numbers,
  even if one doesn’t // know what the investment portfolio is .

void do reaction( std :: vector<std :: size t> &n rates )
 const for ( std :: size t j=0 ; j <
  parameterAffected . size () ; ++j ) std ::
  size t  = parameterAffected [ j ] ; n rates
  [ ratess ] += deltaN[ j ] ; //
   The member defined below is the one that makes //
    this a virtual base class : given the volume and //
     the populations of all the chemical species , //
      derived classes must know how to compute their own rates .

 virtual double rate( const std :: vector<std :: size t> &pop, double Return ) const = 0 ;

}
%-----------------------------------------------------------------------------------
%-----------------------------------------------------------------------------------
```

Implementation of Optimal Investment in C/C++

```
/*
 * %=====================================================================================
 -------------------------------------------------------------------------------------
 *
 *       Filename:  Optimal Strategy Analysis.cpp
 *
 *        Version:  1.0
 *        Created:  2022
 *       Revision:  none
 *       Compiler:  gcc/intel compiler
 *
 *         Author:  John Mutepuwa
 *                  University of Malawi
 *
 * %=====================================================================================
 */

# include <cassert> // etc ... assert ( /something that should be true / ) ;
 In our case we return true values for parameters >=0,
```

```
assert ( Return != 0.0 ) ; rate = 1.0 / Return ;

class Optimization : public Return // X(t) = X(0) + D; X(0) is initial wealth,
 D is partial sum of expenses, consumption and investments.

{ public :
Optimization( double k, std :: size_t x_idx , std :: size_t d_idx )
{
double paraAffected; ReturnDependsOn
rate const = k ; // Install the any rate or parameter
(beta, sigma, mu, r, S_0) that affects X(t)

% Note which parameters/rates the Return depends on

ReturnDependsOn . resize ( 1 ); //Return is X(t) at t
ReturnDependsOn [0] = x_idx;

% Note which parameters/rates the Initial wealth X(0) affects
paraAffected . resize ( 2 );
paraAffected [0] = x_idx ; speciesAffected [1] = d_idx ;

% Note the consequences of the strategy
```

```
sigma . resize (2) ; sigma [0] = 2 ; //volatility
sigma[1] = 1 ;

}

// Computing parameter effects

 double rate( const std :: vector<std :: size t> &pop, double Return ) const

}
{
std :: size t_std :: size t_idx = ReturnDependsOn[0] ; paraAffected = pop[t_idx] ;

}

double rate = 0.0 ; assert ( Return >= 0.0 ) ;

if (paraAffected > 0 ) rate = (rate const / (Return const)) ; rate ==paraAffected;

 return( Return) ;

//plot in cmake gc++
```

```
}
%-------------------------------------------------------------------------------------------------------------------------------------------
%---------------------------------------------------------------------------------------------------------------------
```